Well balancedness and error balance

observations and ideas related to the approximation of (hyperbolic) balance laws

Mario Ricchiuto

team CARDAMOM
Inria Bordeaux Sud-Ouest
France

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

with some initial data

$$u(t = 0, x) = u_0(x)$$

and appropriately prescribed boundary data

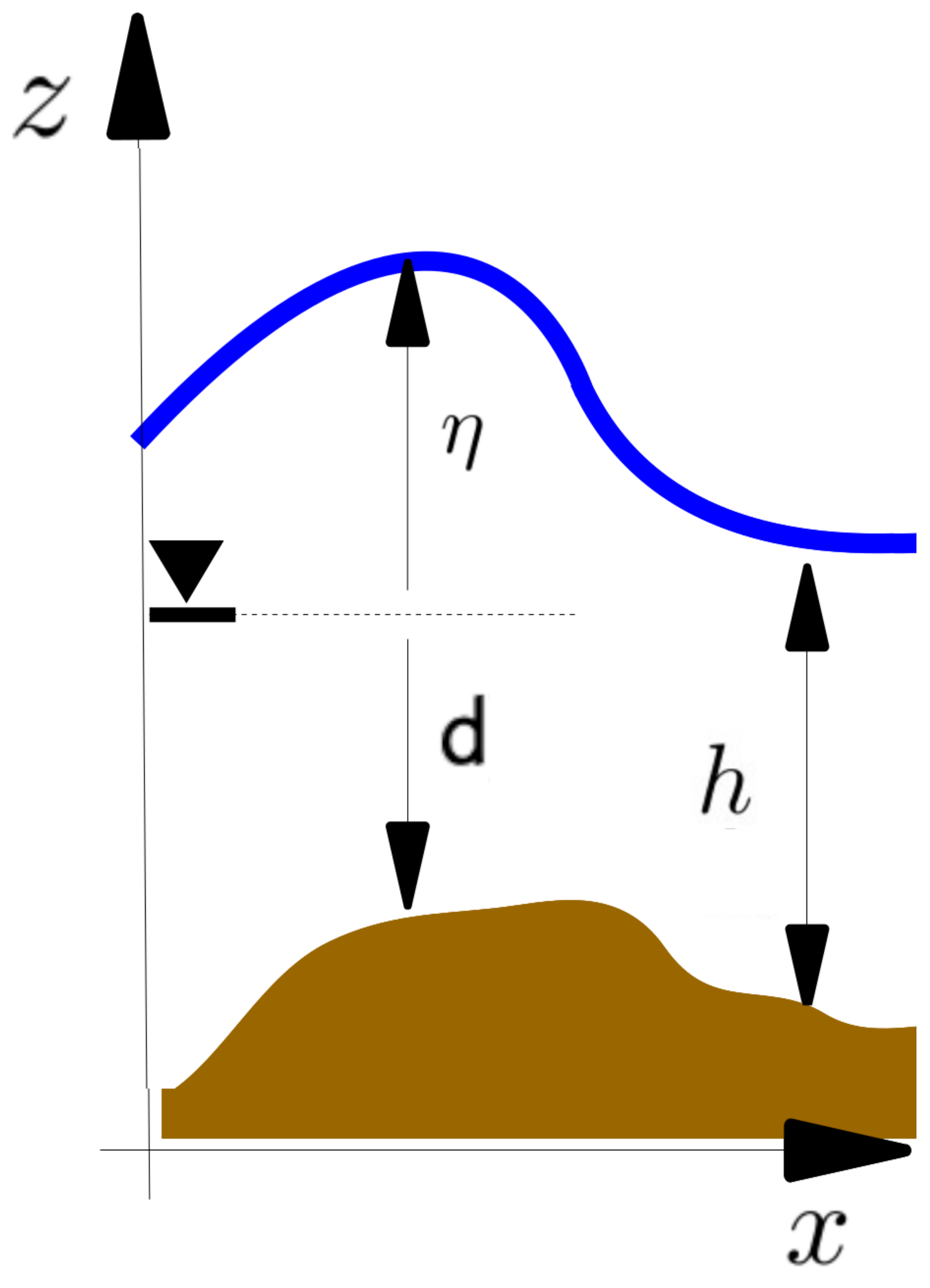
(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

short notation for the source

$$S(u; d) = S(u, \partial_x u, \partial_t u, \text{etc.}; d)$$

external data $d = d(x)$ provided in some form (often not analytically)



(hyperbolic) balance laws

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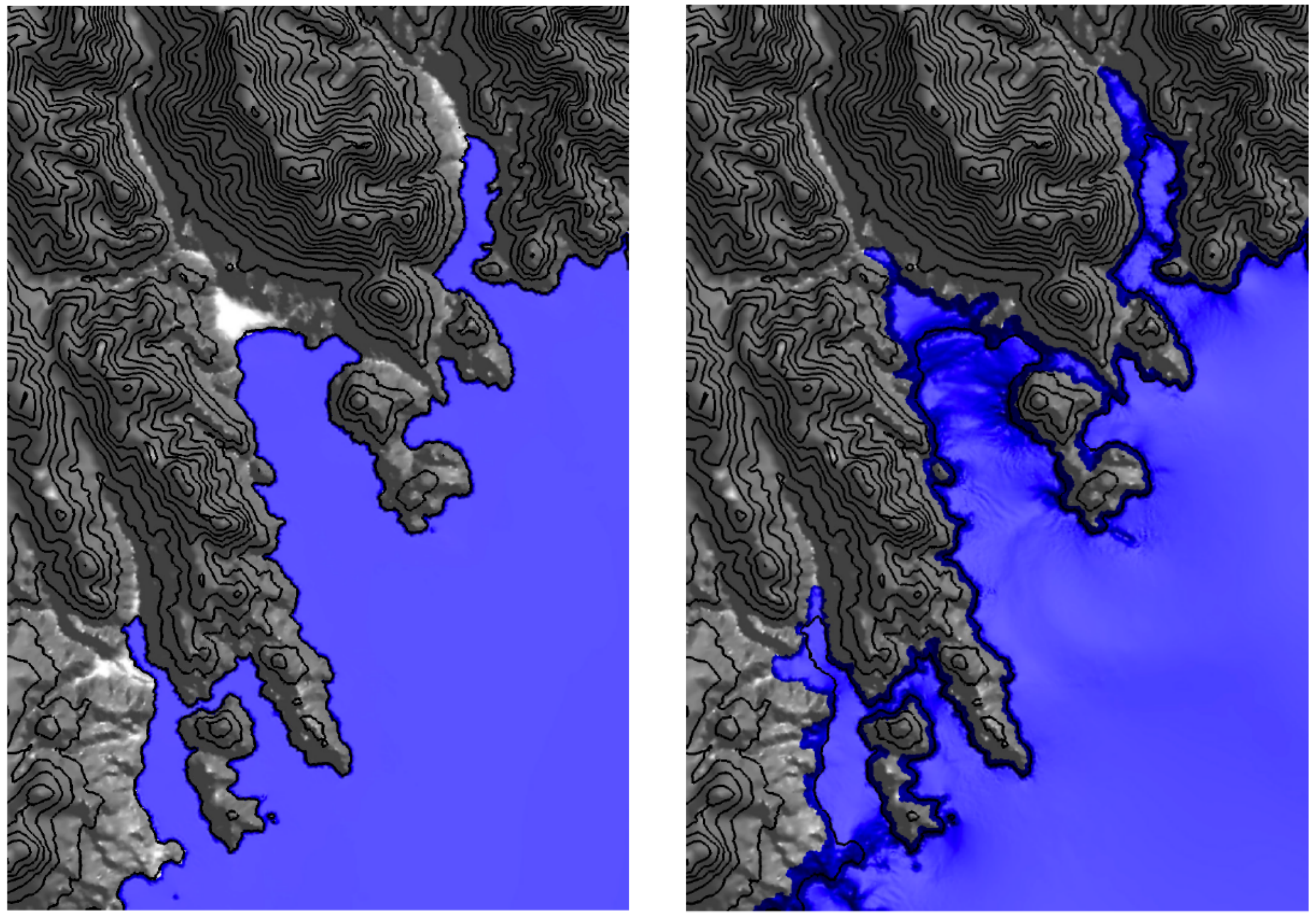

Iwate prefecture (2011 Tohoku tsunami)

L. Arpaia and MR, J.Comput.Phys. 2020

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

hyperbolic balance law

$$A := f_u = R\,\Lambda\,L = R\,\mathrm{diag}(\lambda_j)\,L$$

$$\partial_t \mathscr{U}(u; d) + \partial_x \mathscr{F}(u; d) \leq 0$$

entropy pair/inequality

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

what is consistency for this problem ?

$$f(u) = \text{const} \quad - \quad u = \text{const}$$

no longer admissible states

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

particular (un)steady states

$$\partial_x f(u) + S(u; d) = 0$$

For example

$$v(u; d) = \text{const}$$

steady invariants

$$f(x) = f_0 - \int_{x_0}^{x} S(u; d(s))ds$$

equilibrium flux distribution
(more or less explicitly computable)

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

particular (un)steady states

$$\partial_x f(u) + S(u; d) = 0$$

For example

$$S(u; d) = S(\partial_t u, \text{ etc.})$$

propagating waves.

Example: Sw + dispersion

$$S = -\gamma \partial_{xxt} u$$

exact solitary waves

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(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

what is consistency for this problem ?

Well-Balanced and related questions:

- natural way of expressing consistency for balance laws
- most basic formulation generalised notion of consistency wrt constants
- one problem is: what should be constant here ?
- another problem is : how about multi D ?

(hyperbolic) balance laws

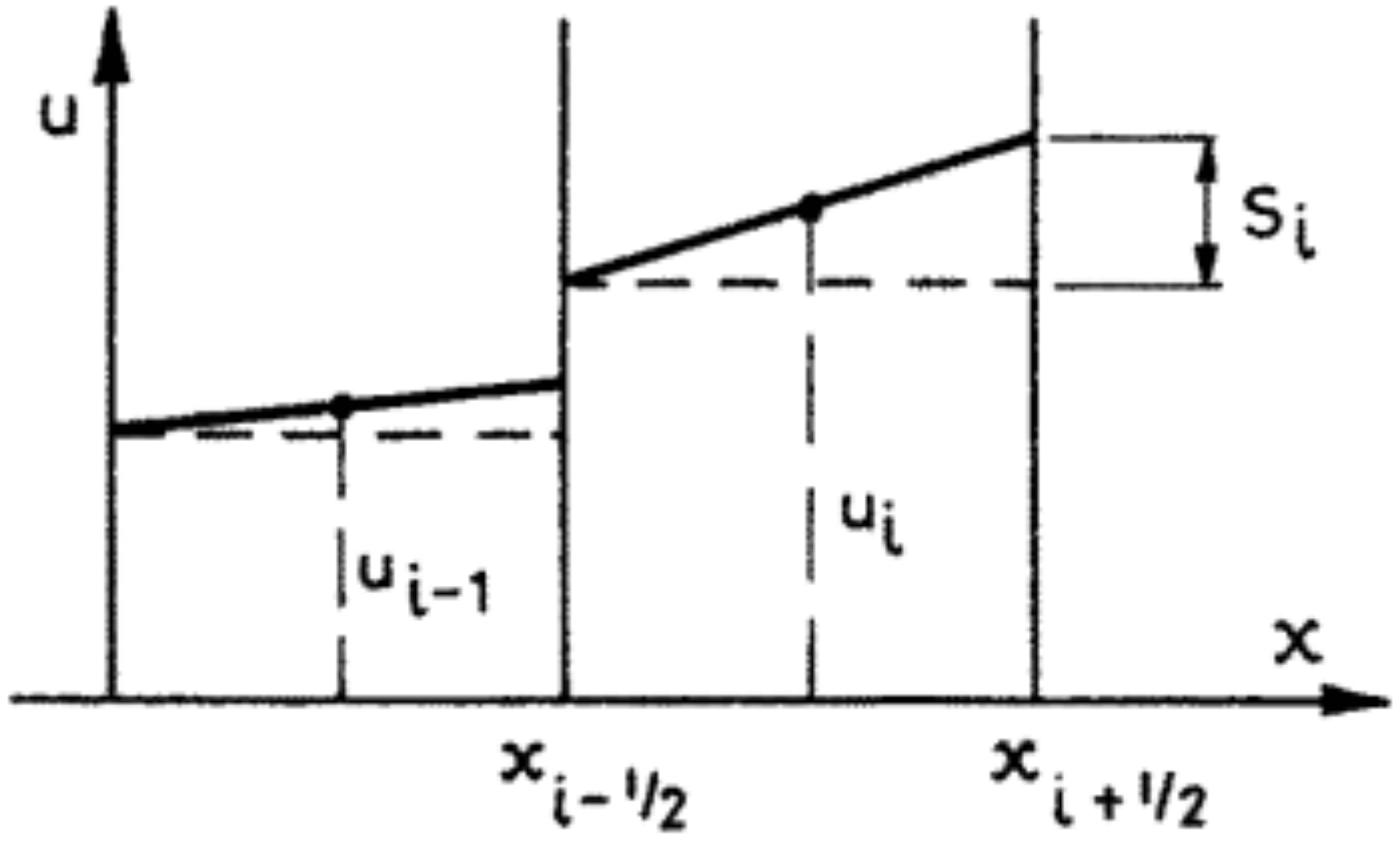
$$\partial_t u + a\partial_x u - q(x) = 0\,, \quad a > 0$$

Once upon a time … :

P.L. Roe. Upwind differencing schemes for hyperbolic conservation laws with source terms. In Claude Carasso, Denis Serre, and Pierre-Arnaud Raviart, editors, *Nonlinear Hyperbolic Problems*, pages 41–51, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

(hyperbolic) balance laws

$$\partial_t u + a\partial_x u - q(x) = 0\,, \quad a > 0$$



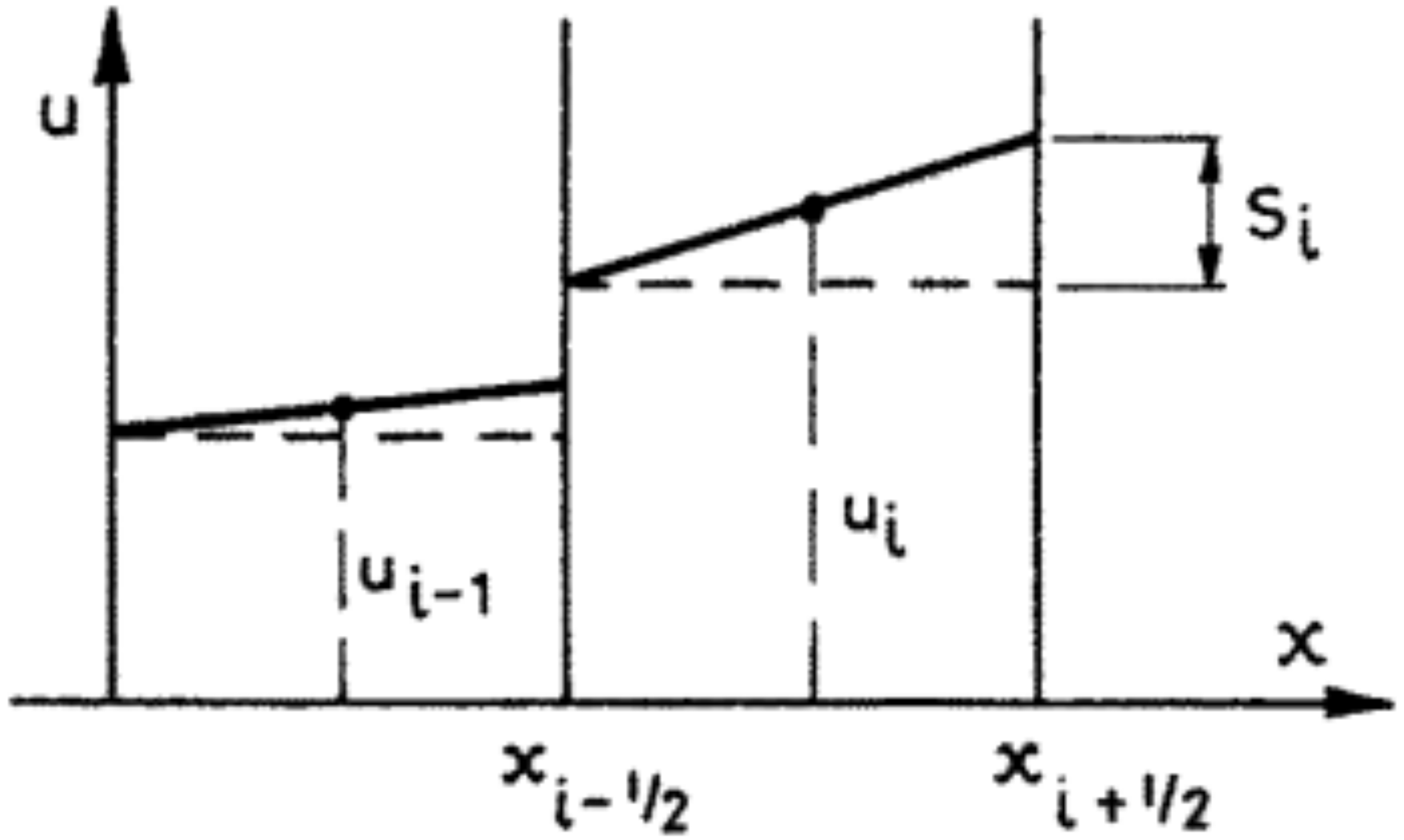
Once upon a time … :

$$u_i^{n+1} = u_i^n - \nu(u_i^n - u_{i-1}^n) + \tfrac{1}{2}\nu(1 - \nu)S_{i-1} - \tfrac{1}{2}\nu(1 - \nu)S_i$$
$$+ [(1 - \tfrac{1}{2}\nu)q_i + \tfrac{1}{2}\nu q_{i-1}]\Delta t$$

Upwind fluxes, source integration along characteristics, piecewise linear data, constant source

(hyperbolic) balance laws

$$\partial_t u + a\partial_x u - q(x) = 0\,, \quad a > 0$$



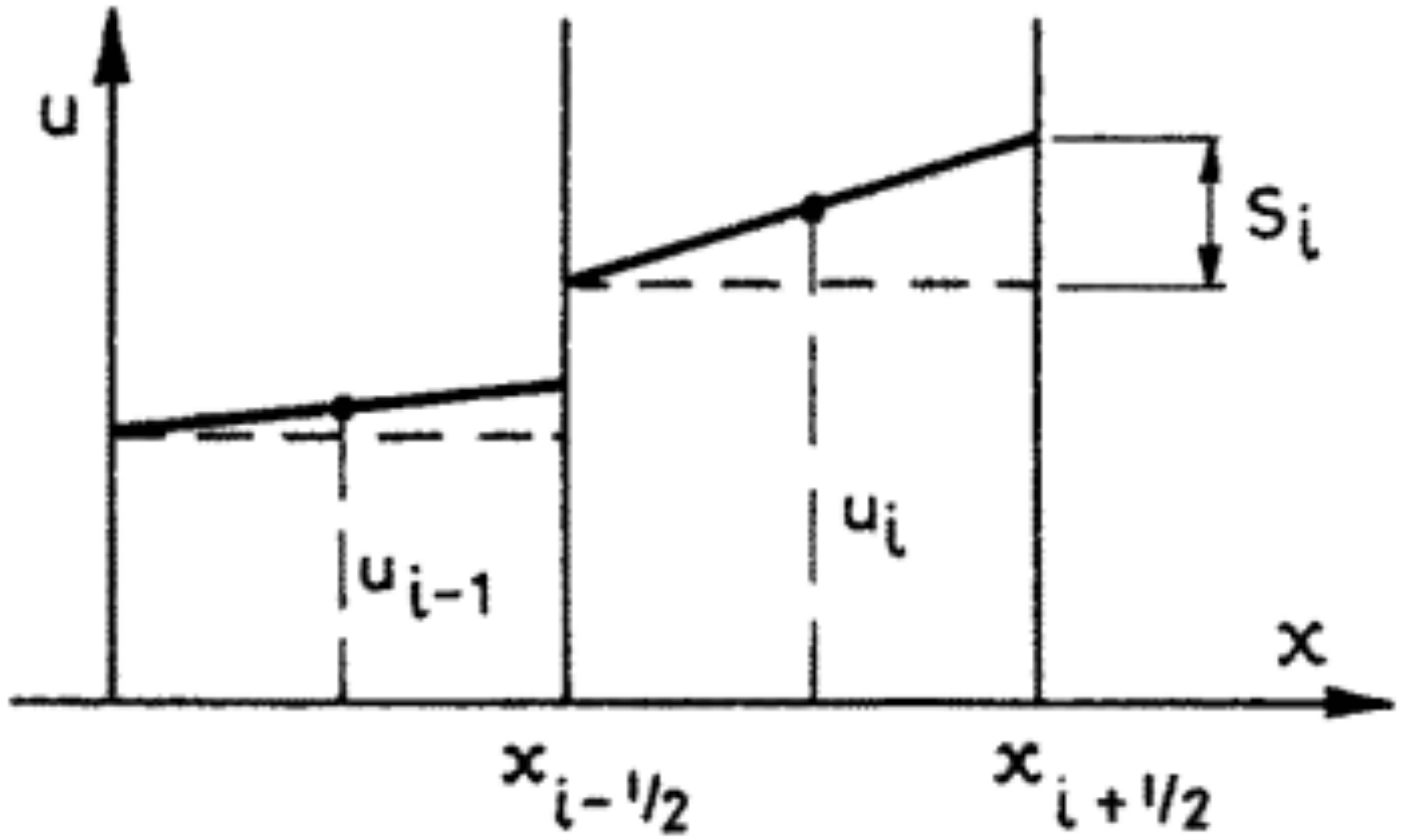
Once upon a time … :

several authors [6,7,8] have felt the attraction of considering data which is in piecewise equilibrium. That is, the data is projected into a representation such that the steady flow equations are satisfied within each cell. In our simple model equation, that means choosing

$$S_i = \frac{q_i \Delta x}{a} = \frac{q_i \Delta t}{\nu} \qquad (15)$$

(hyperbolic) balance laws

$$\partial_t u + a\partial_x u - q(x) = 0\,, \quad a > 0$$



Once upon a time … :

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\phi_{i-\frac{1}{2}}$$

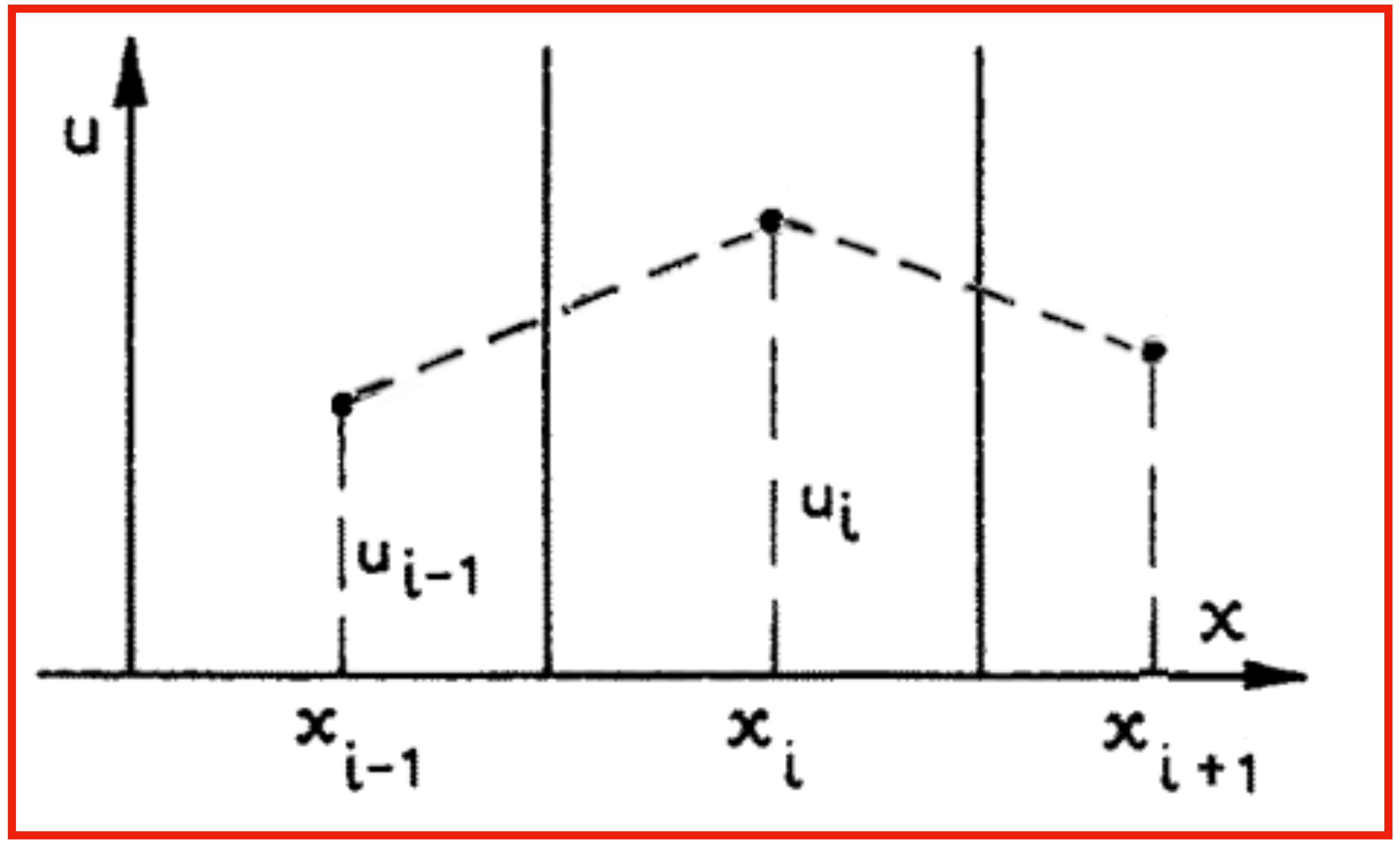
we can measure the extent to which they are out of equilibrium (with each other now, now internally) by the quantity

$$\phi_{i-\frac{1}{2}} = a(u_i - u_{i-1}) - \tfrac{1}{2}\Delta x(q_{i-1} + q_i) \quad (17)$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Residual distribution/fluctuation splitting:



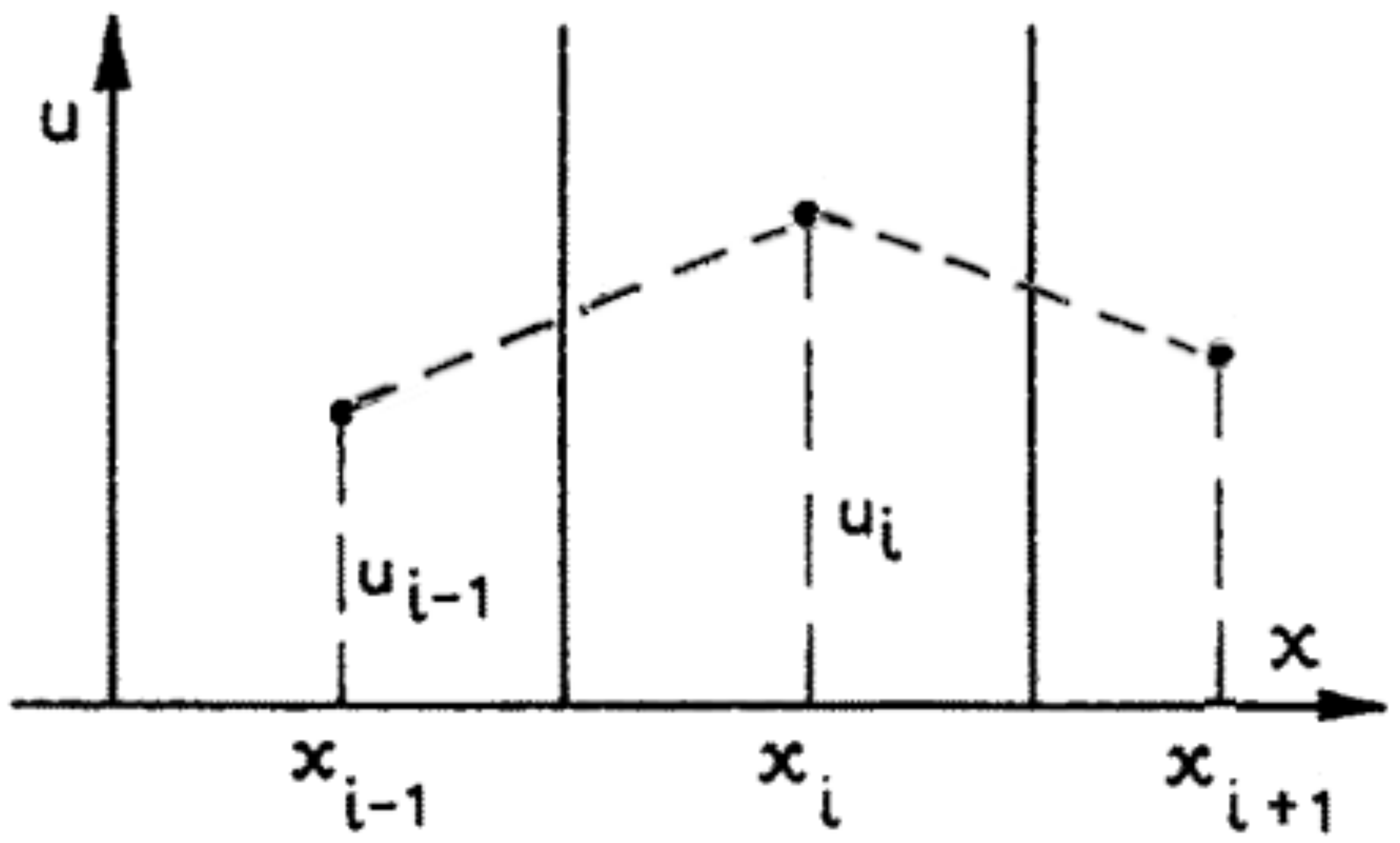
$$\boxed{\phi^{i\pm1/2} = \int_{\Delta_{i\pm1/2}x} (\partial_x f + S)}$$

continuous approximation

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\max(a,0)}{a}\phi^{i-1/2} + \frac{\min(a,0)}{a}\phi^{i+1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u;d) = 0$$



Residual distribution/fluctuation splitting:

$$\phi^{i\pm 1/2} = \int_{\Delta_{i\pm 1/2}x} (\partial_x f + S)$$

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2}$$

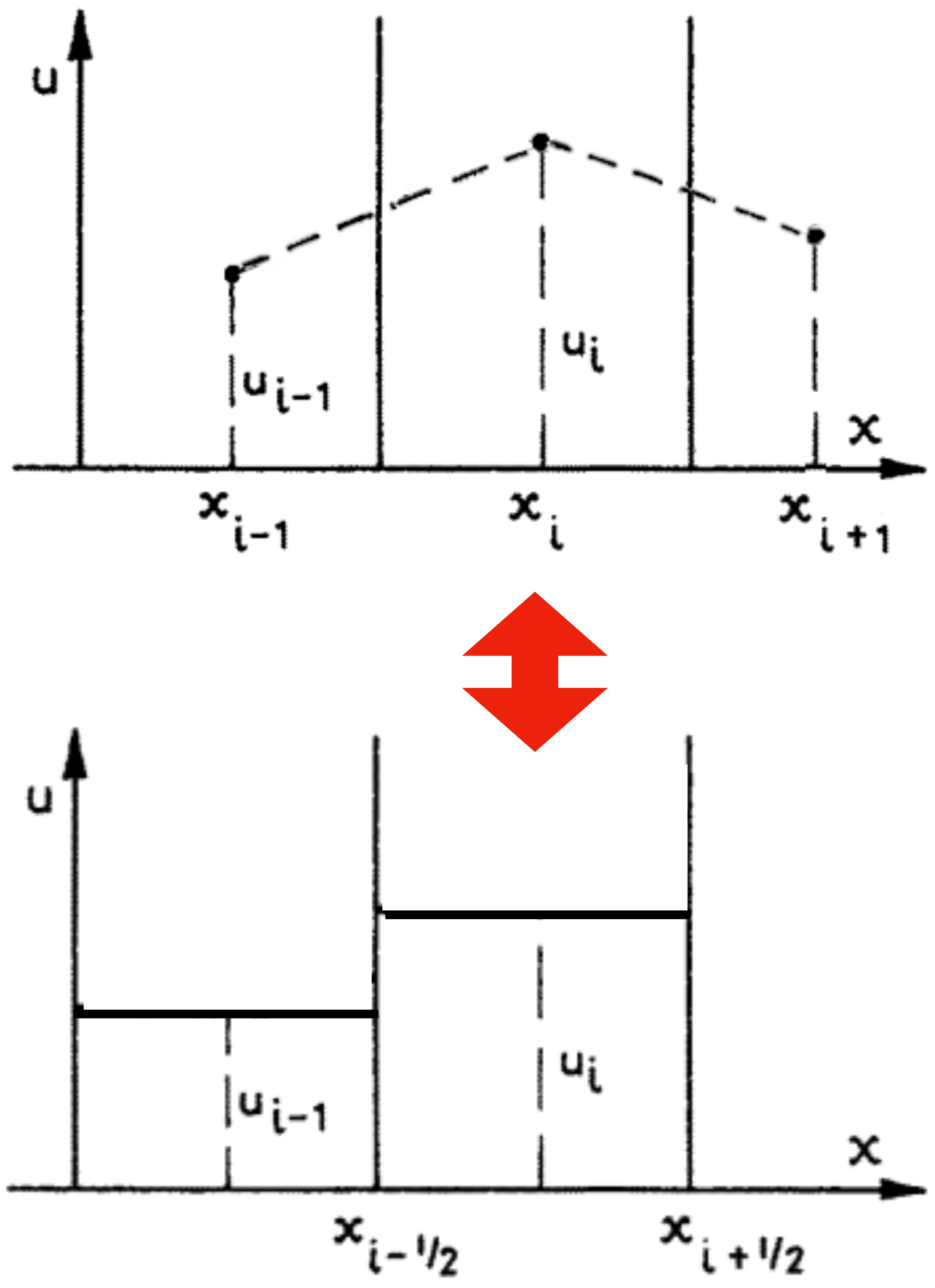
distribution/splitting

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Consistent fluxes



(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Consistent fluxes

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} := f_{i+1} - f_i$$

$$\hat{f}_{i+1/2} = f_i + \phi_i^{i+1/2} = f_{i+1} - \phi_{i+1}^{i+1/2}$$

⇕

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{f}_{i+1/2} - \hat{f}_{i-1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

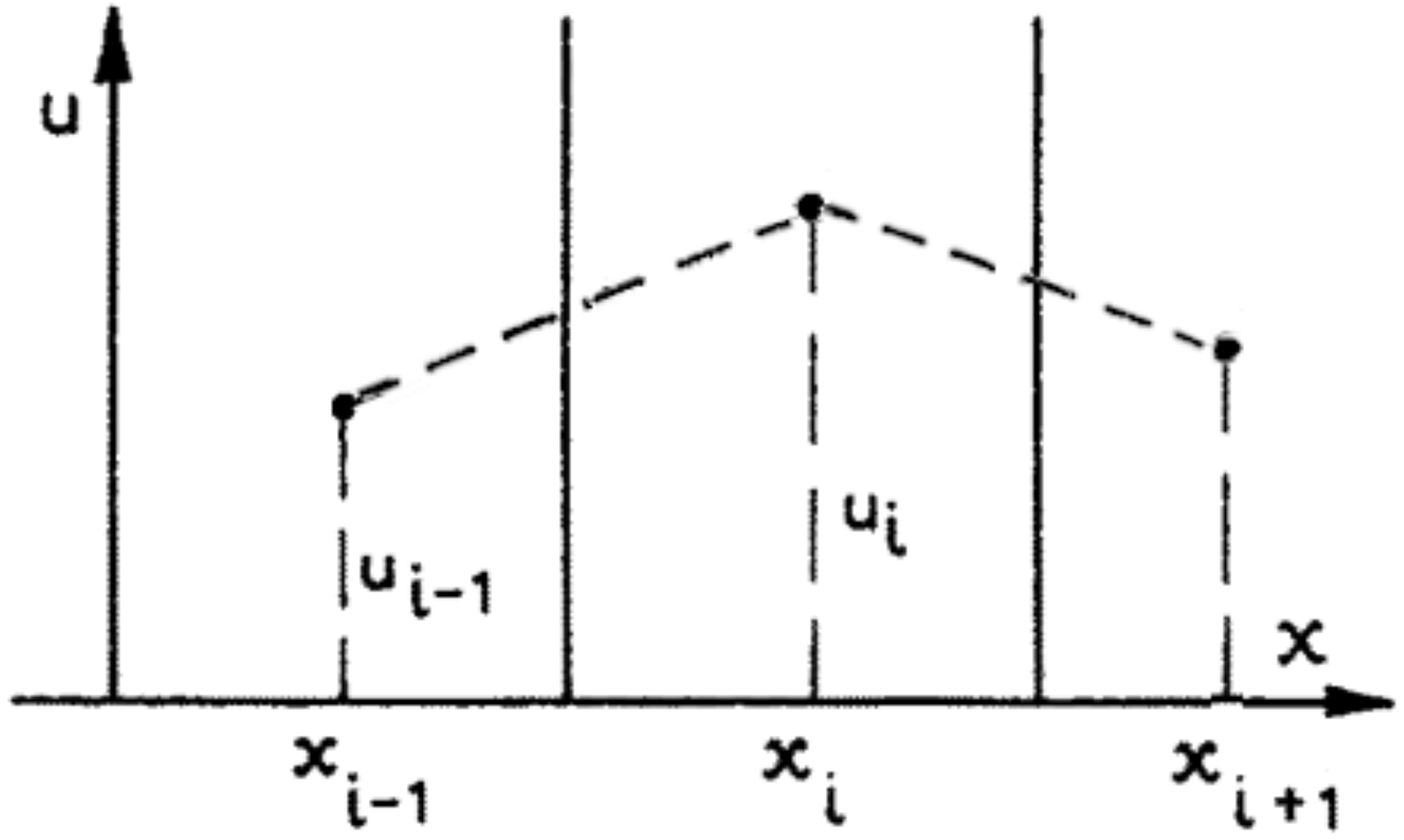
Consistent fluxes

Notion of consistency wrt constants

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{f}_{i+1/2} - \hat{f}_{i-1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$



Residual distribution/fluctuation splitting:

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

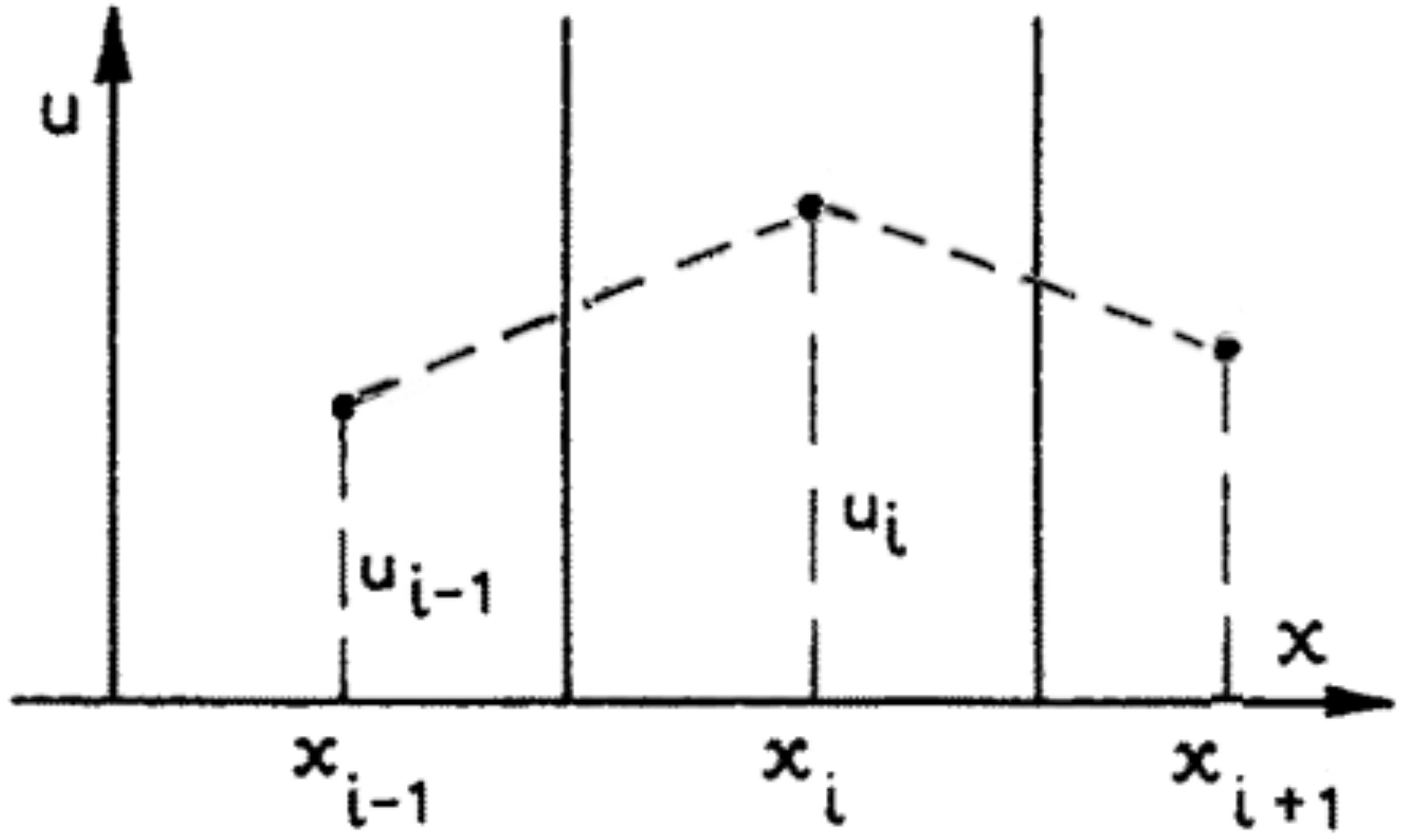
$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2}$$

$$\phi^{i\pm 1/2} = \int_{\Delta_{i\pm 1/2} x} (\partial_x f + S)$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Residual distribution and global fluxes



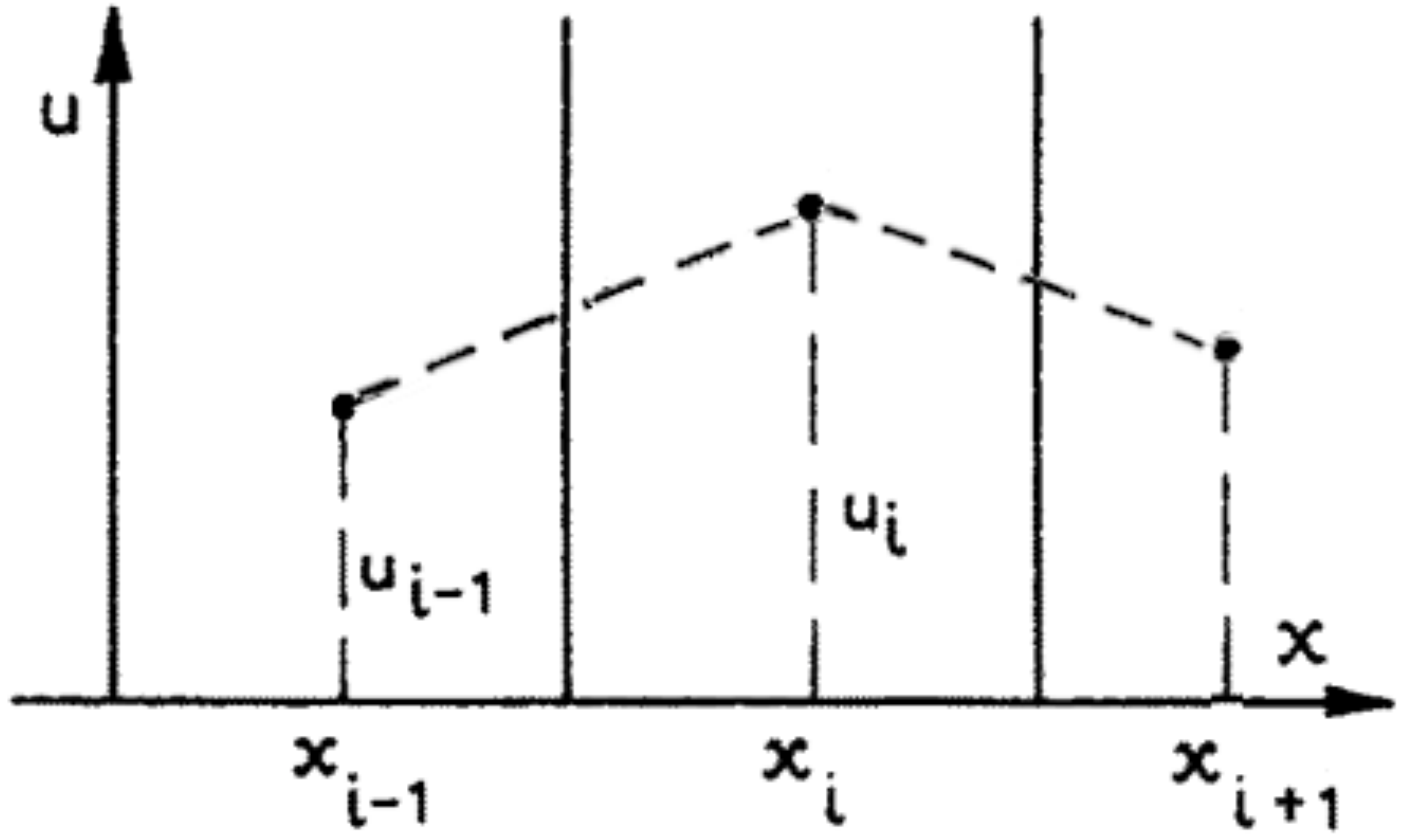
$$S_{i\pm1/2} := \frac{1}{\Delta_{i\pm1/2}x} \int_{\Delta_{i\pm1/2}x} S(u(x)); d(x))$$

$$s_i := s_{i-1} + \Delta_{i-1/2}x \; S_{i-1/2}$$

$$g_i := f_i + s_i$$ global flux

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$



Residual distribution and global fluxes

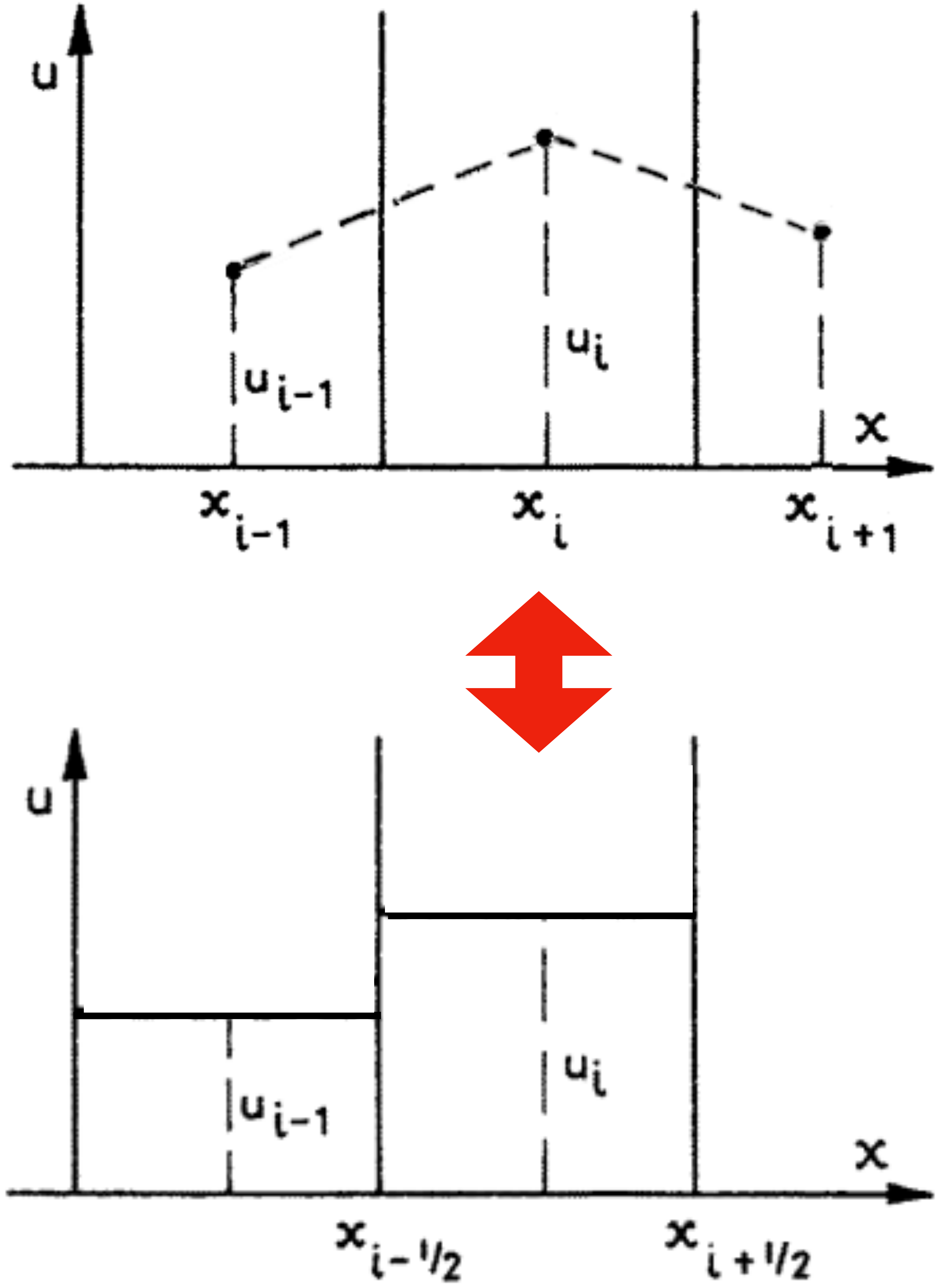
$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} = \boxed{\int_{\Delta_{i+1/2}x} (\partial_x f + S) = \Delta_{i+1/2} g = \int_{\Delta_{i+1/2}x} \partial_x g}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Consistent global fluxes



(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u;d) = 0$$

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Consistent global fluxes

$$\phi_i^{i-1/2} + \phi_{i-1}^{i-1/2} = \phi^{i+1/2} := f_{i+1} - f_i + \Delta_{i+1/2}x\; S_{i+1/2}$$
$$= g_{i+1} - g_i$$

$$\hat{g}_{i+1/2} = g_i + \phi_i^{i+1/2} = g_{i+1} - \phi_{i+1}^{i+1/2}$$

⇕

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Consistent global fluxes

$$\Updownarrow$$

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

(surrogate) notion of consistency:
consistency wrt constant global fluxes

no info on the structure of the solution
(in the source quadrature, cf later)

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Consistent global fluxes: work fine

Delis and Katsaounis, IJNMF 2003
Chertock et al, J.Comput.Phys. 2018
Cheng et al, J.Sci.Comp. 2019

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0$$

Residual distribution

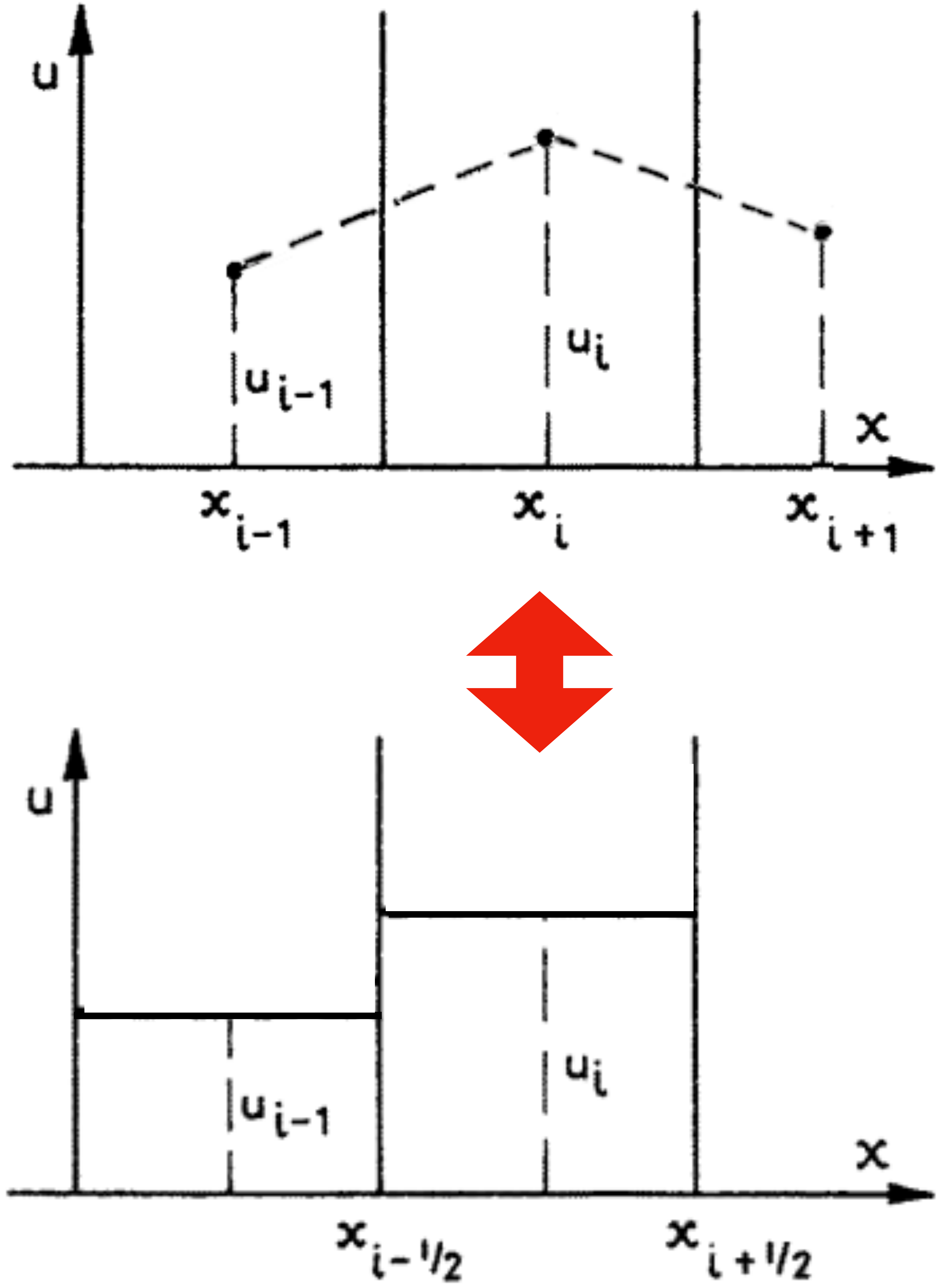
Roe, 1987
Castro, Pares and co. : Well-Balanced/path conservative
Karni, Hernandez-Duenas, Balbas
RD stuff in 1D: Abgrall, MR

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

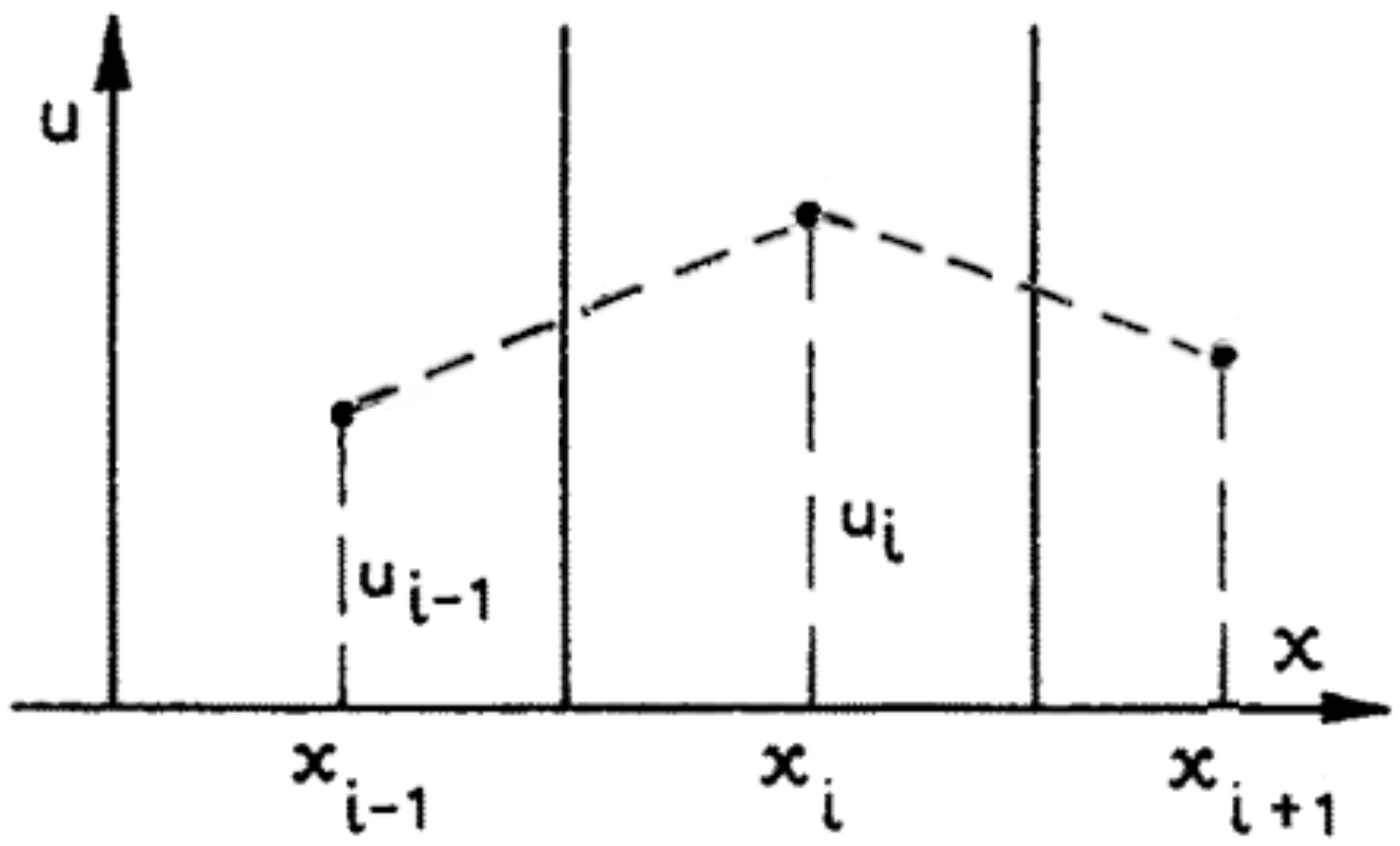
Consistent global fluxes

First order accurate !



(hyperbolic) balance laws

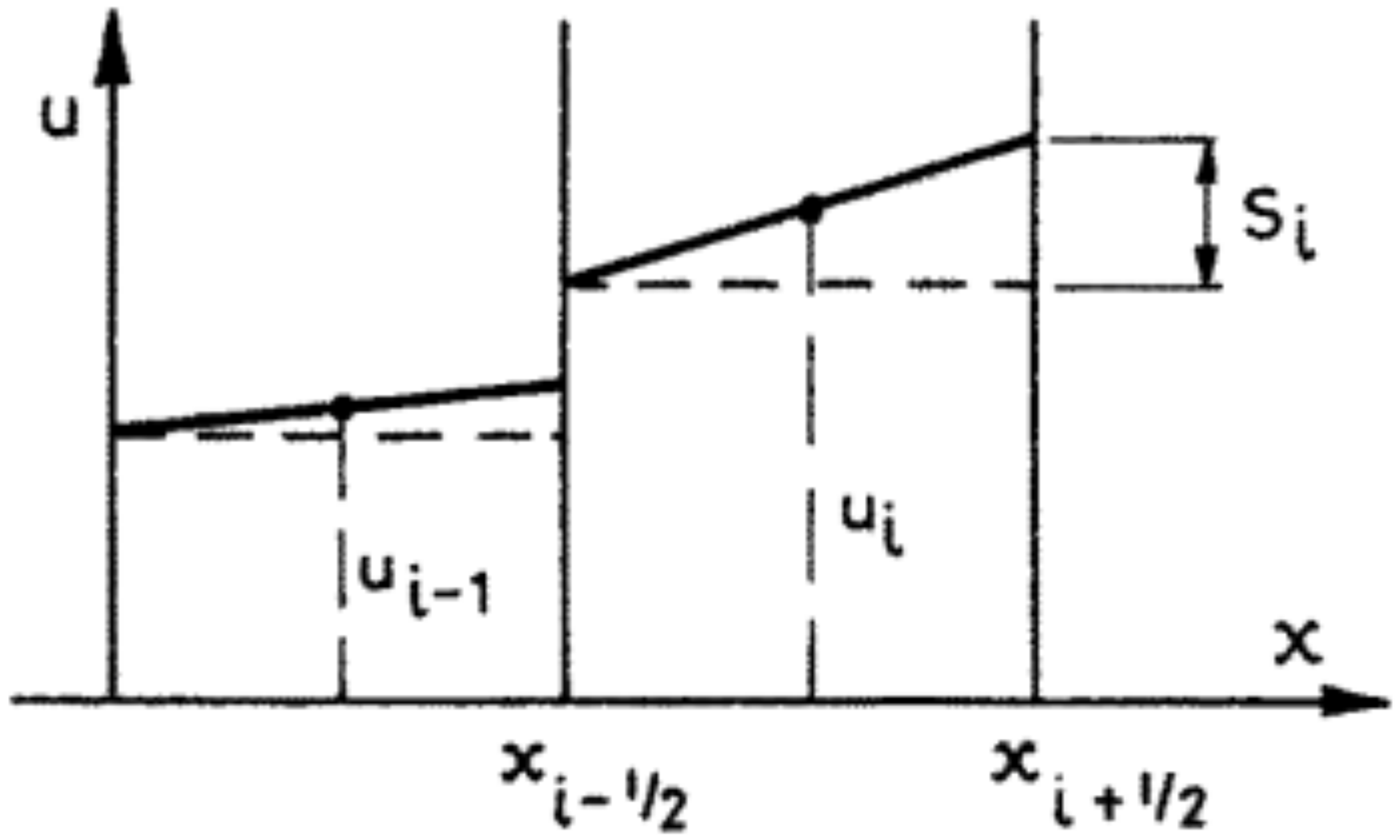
$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$



Consistent global fluxes + reconstruction

Delis and Katsaounis, IJNMF 2003
Chertock et al, J.Comput.Phys. 2018
Cheng et al, J.Sci.Comp. 2019

cf. talk by Alina



(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

particular (un)steady states

$$\partial_x f(u) + S(u; d) = 0$$

$$S(u; d) = S(\partial_t u,\ \text{etc.})$$

propagating waves.

Example: Sw + dispersion

$$S = -\gamma \partial_{xxt} u$$

exact solitary waves

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

particular (un)steady states

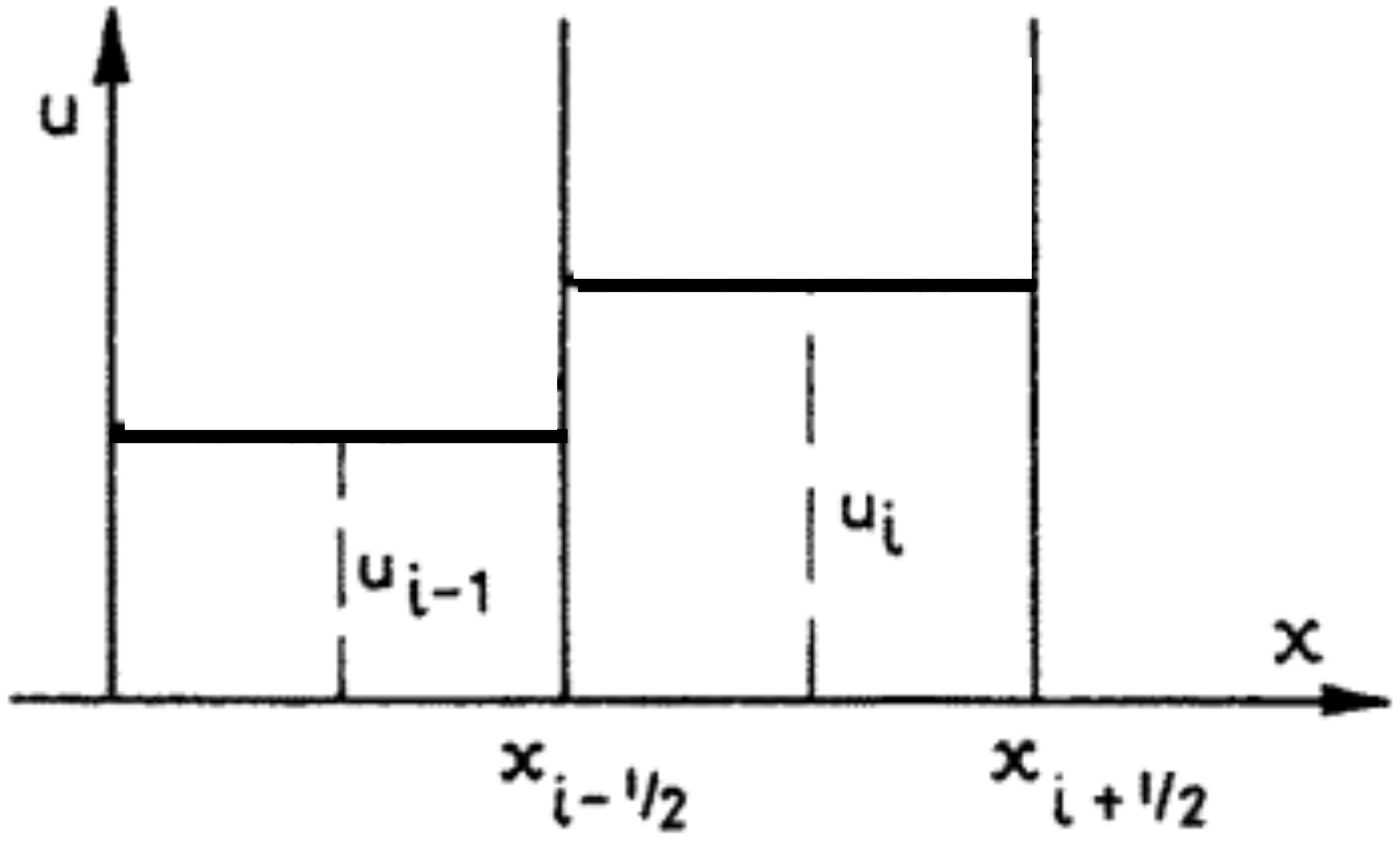
$$\mathcal{S}(u; d) = \partial_t u$$

propagating waves…

$$\partial_x f(u) + \mathcal{S}(u; d) = 0$$

(hyperbolic) balance laws

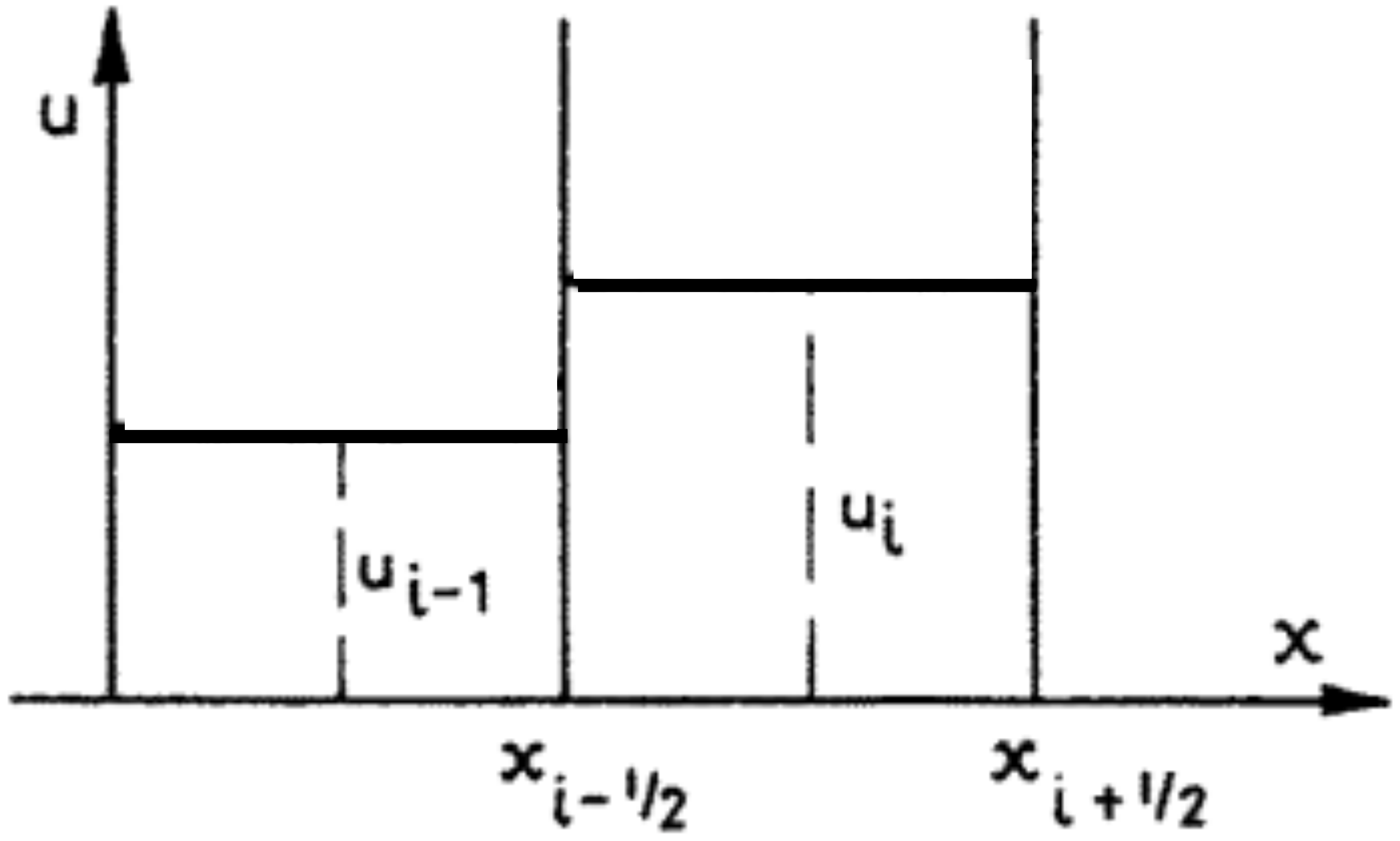
$$\mathcal{S}(u) + \partial_x \mathrm{f}(\mathrm{u}) = 0$$



$$\hat{g}_{\mathrm{i}+1/2} - \hat{g}_{\mathrm{i}-1/2} = 0$$

(hyperbolic) balance laws

$$\mathcal{S}(u) + \partial_x \mathrm{f}(u) = 0$$



Upwind (global) flux

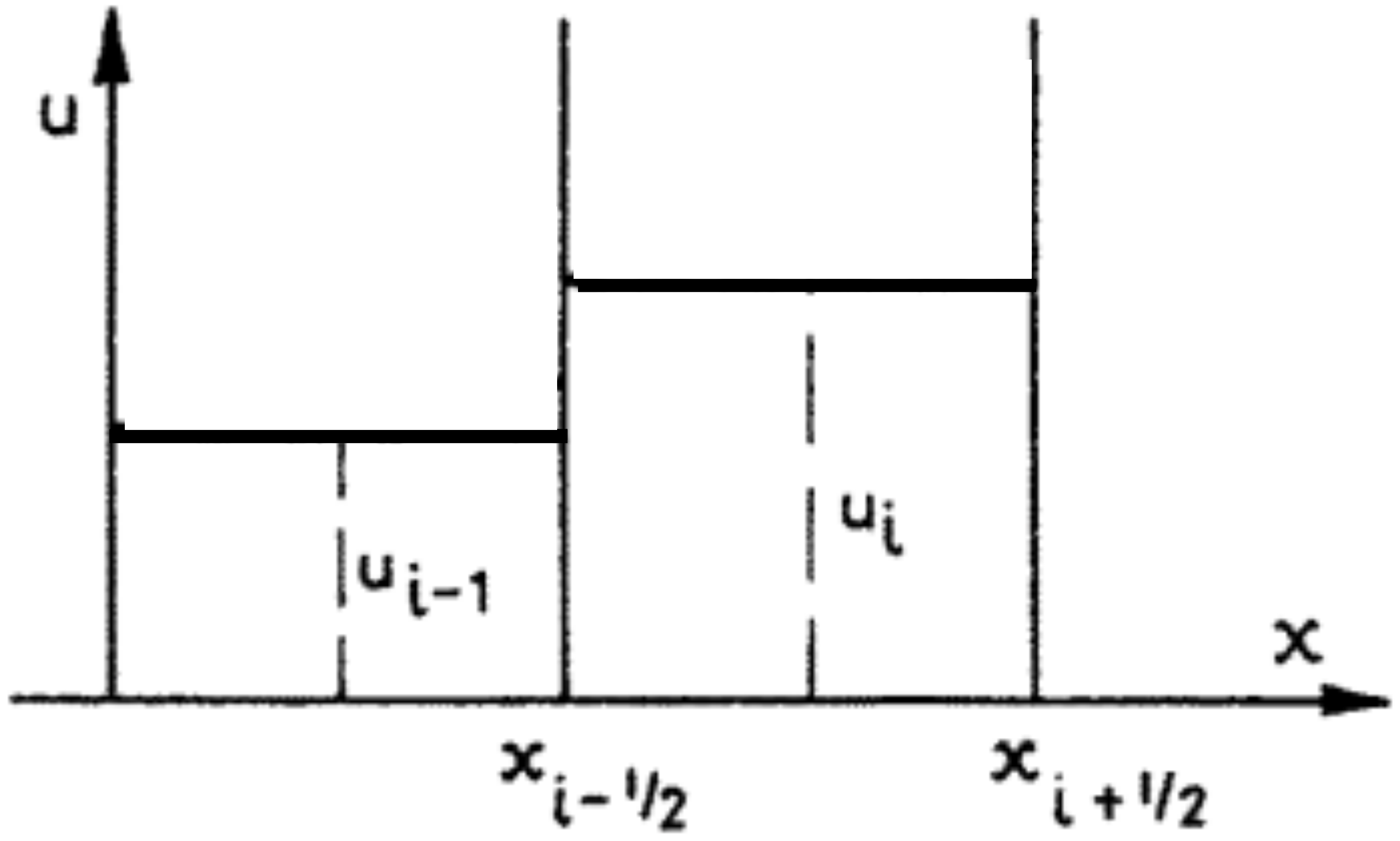
$$\hat{g}^u_{i+1/2} = \frac{g_{i+1} + g_i}{2} - \frac{\mathrm{sign}(A)_{i+1/2}}{2}(g_{i+1} - g_i)$$

$$g_{i+1} = \mathrm{f}_{i+1} + s_{i+1} = \mathrm{f}_{i+1} + s_i + \Delta_{i+1/2}x\, \mathcal{S}_{i+1/2}$$

$$\hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

(hyperbolic) balance laws

$$\mathcal{S}(u) + \partial_x f(u) = 0$$



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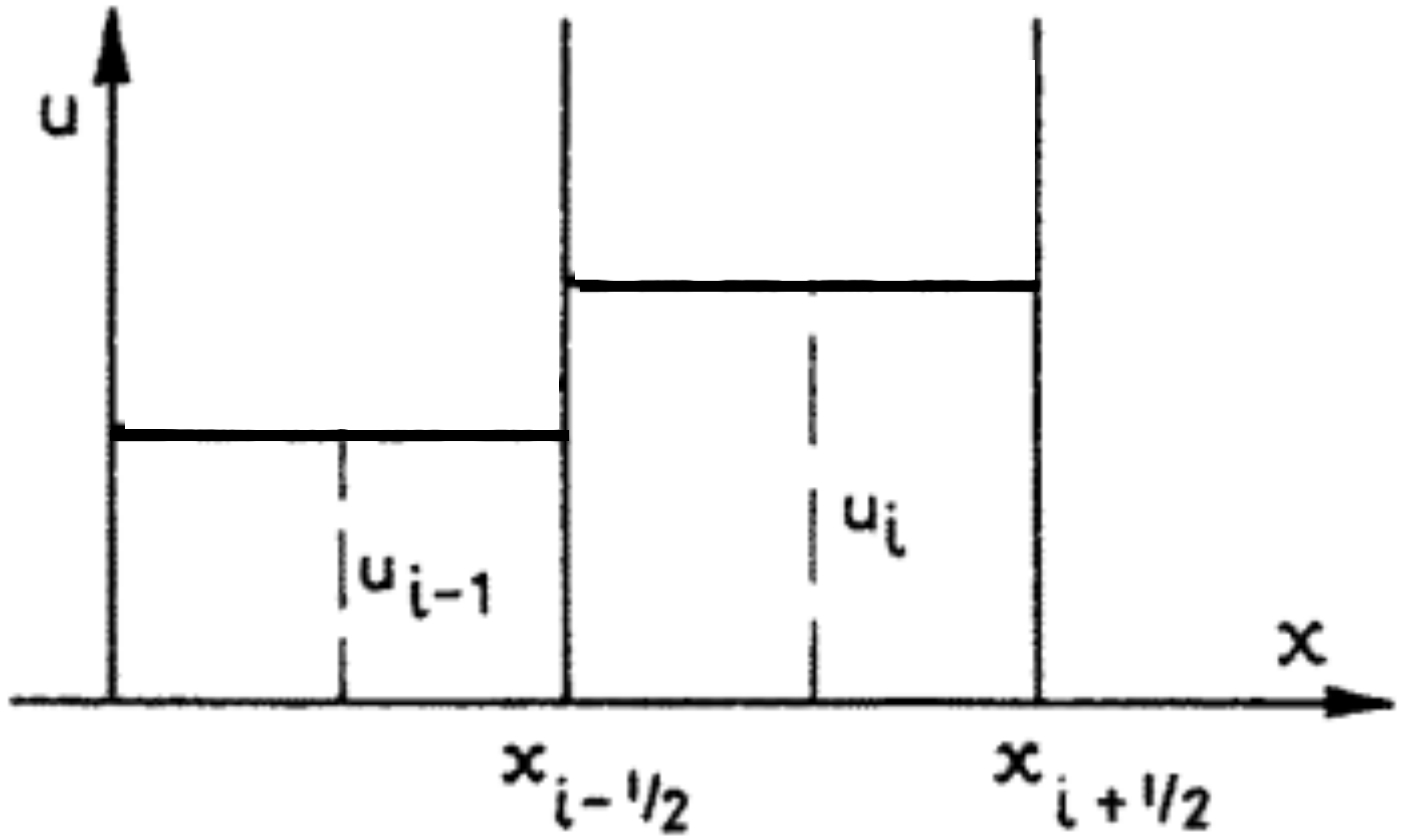
$$g_{i+1} = f_{i+1} + s_{i+1} = f_{i+1} + s_i + \Delta_{i+1/2}x\ \mathcal{S}_{i+1/2}$$

$$\hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$



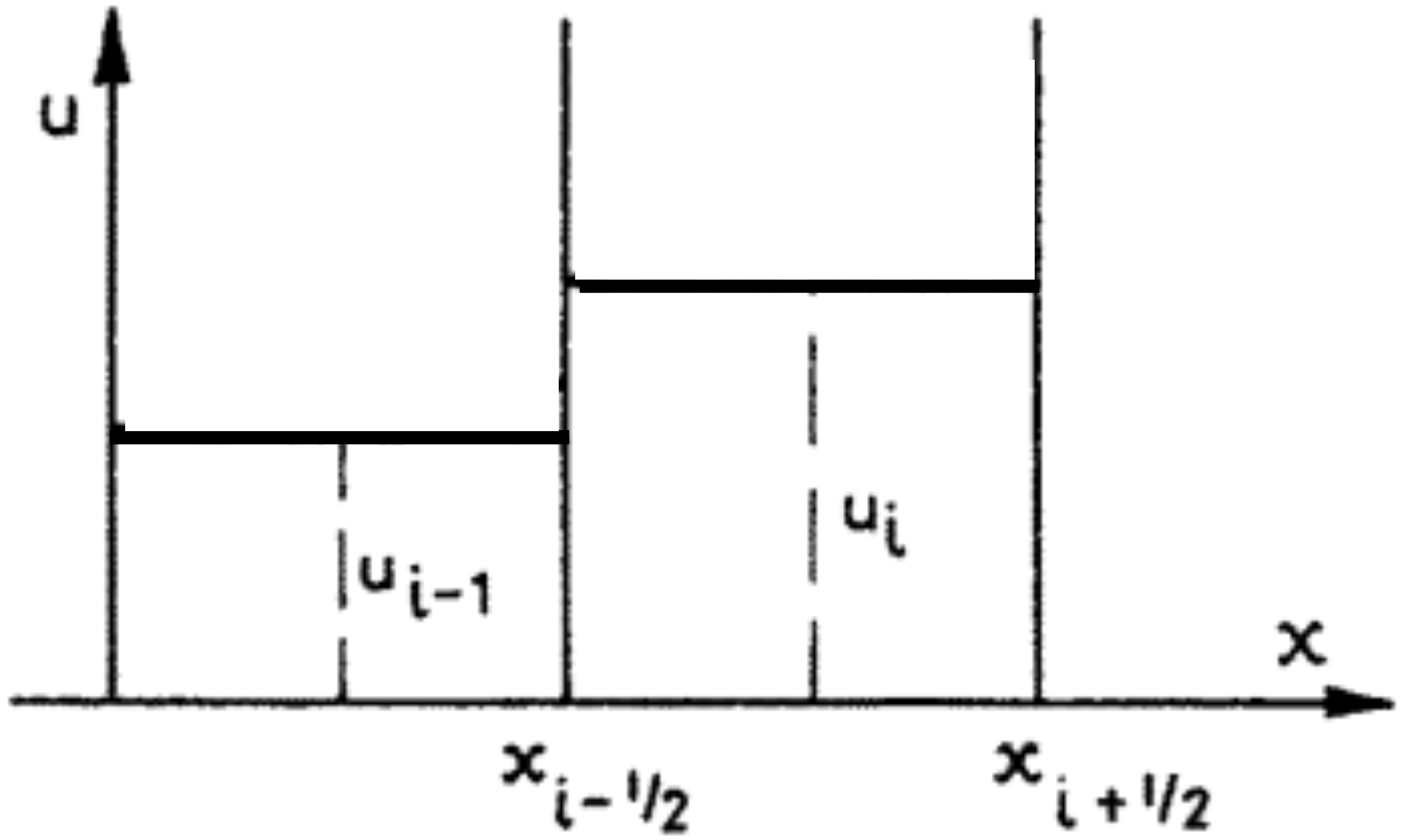
Upwind (global) flux

$$\Delta_i x \frac{d\hat{u}_i^u}{dt} + \hat{f}^u_{i+1/2} - \hat{f}^u_{i-1/2} = 0$$

$$\hat{f}^u_{i+1/2} = \frac{f_{i+1} + f_i}{2} - \frac{\text{sign}(A)_{i+1/2}}{2}(f_{i+1} - f_i)$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$



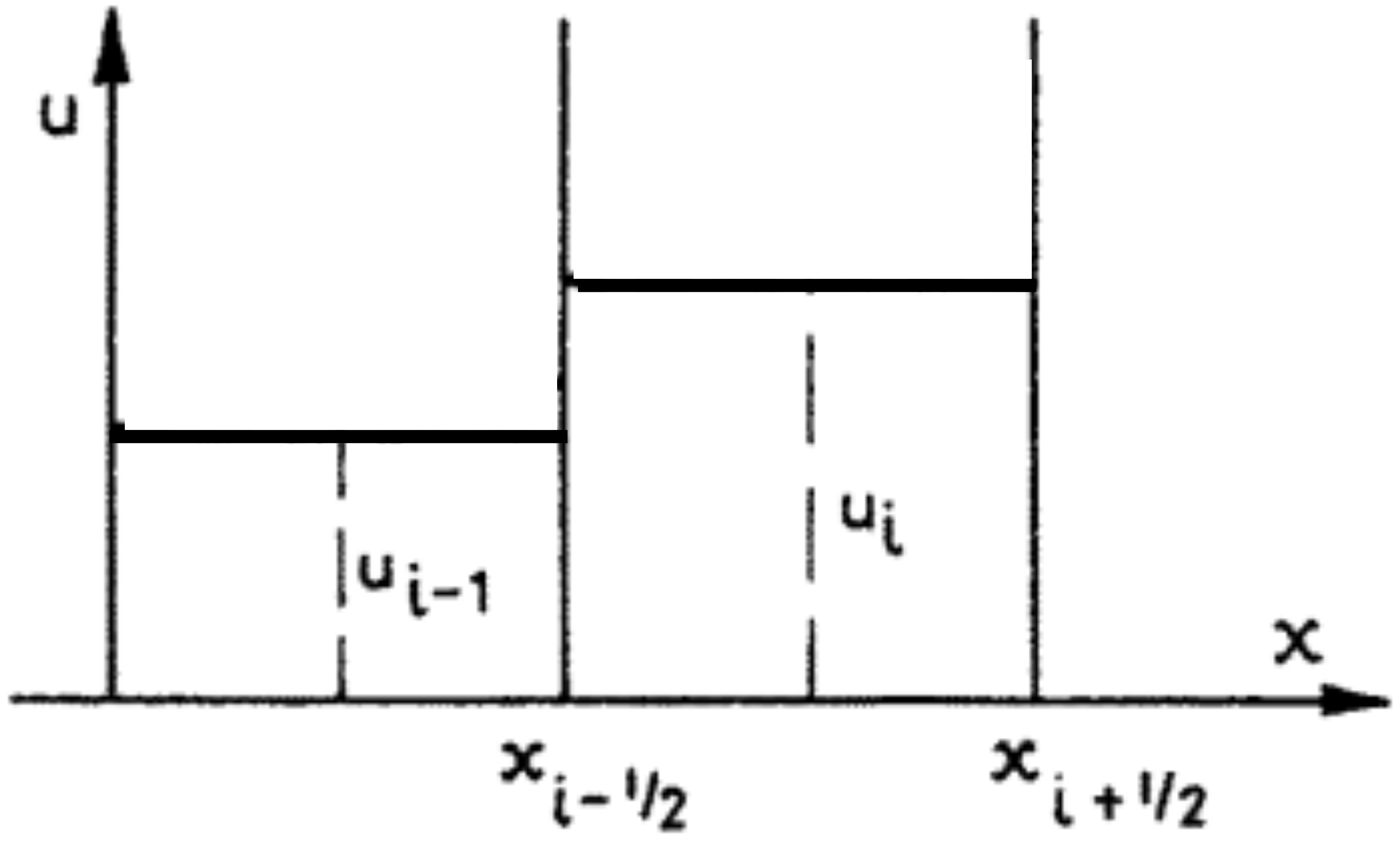
Upwind (global) flux

$$\Delta_i x \frac{d\hat{u}_i^u}{dt} + \hat{f}^u_{i+1/2} - \hat{f}^u_{i-1/2} = 0$$

$$\Delta_i x \frac{d\hat{u}_i^u}{dt} := \Delta_i x \frac{du_i}{dt} + \left\{ \frac{\Delta x}{2} \left(\frac{1}{2} \Delta \frac{du}{dt} + \mathrm{sign}(A) \frac{du}{dt} \right) \right\}_{i+1/2} - \left\{ \frac{\Delta x}{2} \left(\frac{1}{2} \Delta \frac{du}{dt} + \mathrm{sign}(A) \frac{du}{dt} \right) \right\}_{i-1/2}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$



Upwind (global) flux

$$\Delta_i x \frac{d\hat{u}_i^{u}}{dt} + \hat{f}_{i+1/2}^{u} - \hat{f}_{i-1/2}^{u} = 0$$

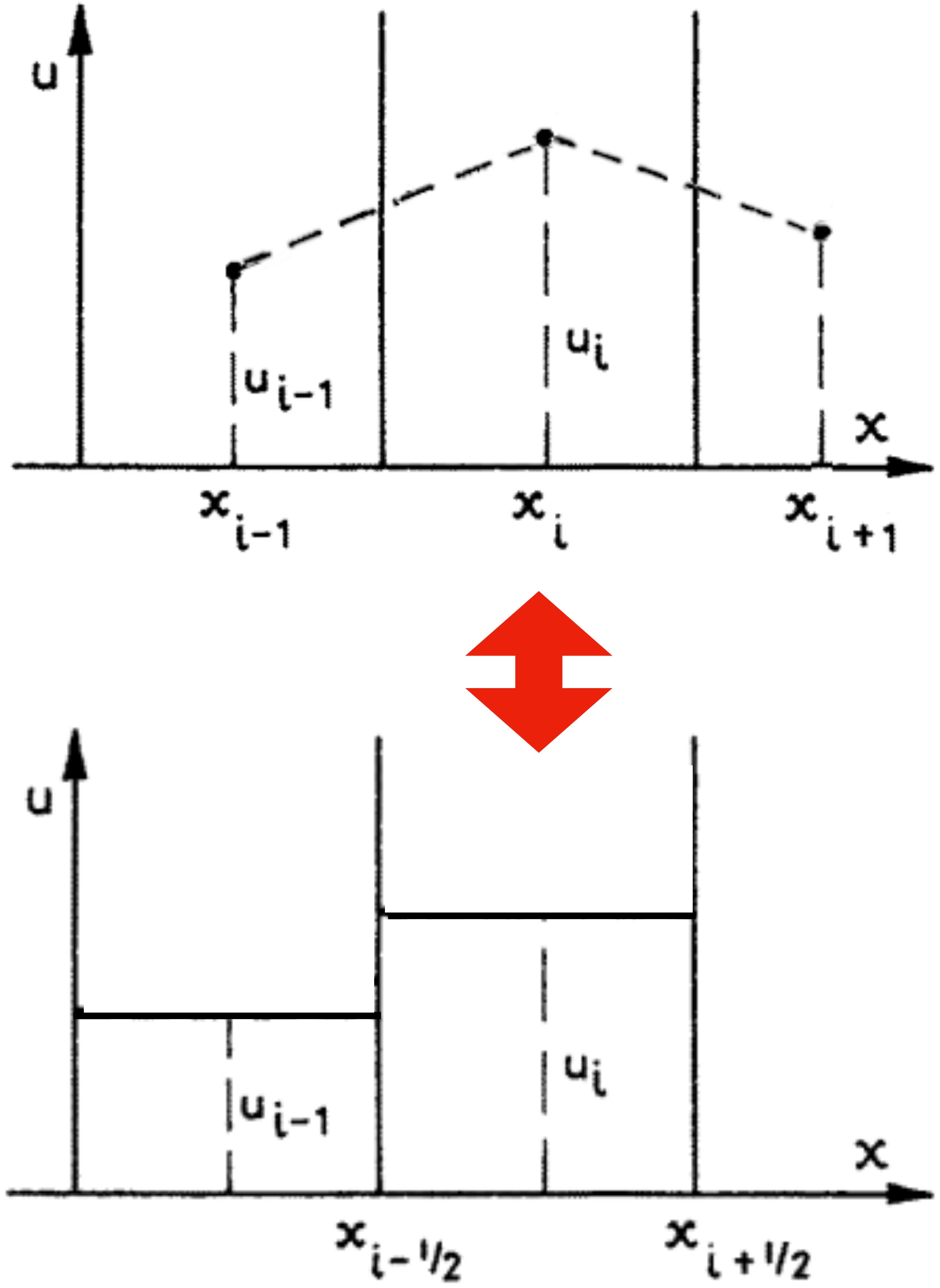
Formal order of accuracy Δx^2 (e.g. by truncated Taylor series)

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Consistent global fluxes: uFEM/SUPG

$$\int (\varphi_i + A \nabla \varphi_i \tau)\left\{\partial_t u + \partial_x f(u)\right\} = 0$$



(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Consistent global fluxes: uFEM/SUPG

$$\int (\varphi_i + A \nabla \varphi_i \tau)\left\{\partial_t u + \partial_x f(u)\right\} = 0$$

$$\phi_i^{i\pm 1/2} = \int_{\Delta_{i\pm 1/2}x} (\varphi_i + A \nabla \varphi_i \tau)\left\{\partial_t u + \partial_x f(u)\right\}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

$$\tau := \frac{\Delta x}{2} |A|^{-1}$$

Linear expansion for f

Consistent global fluxes: uFEM/SUPG

$$\int (\varphi_i + A \nabla \varphi_i \tau)\left\{\partial_t u + \partial_x f(u)\right\} = 0$$

$$\int_{\Delta_{i\pm1/2}x} (\varphi_i + A \nabla \varphi_i \tau)\partial_x f(u) \approx \left(\frac{1}{2} \mp \frac{\text{sign}(A_{i\pm1/2})}{2}\right) \Delta_{i\pm1/2} f$$

(hyperbolic) balance laws

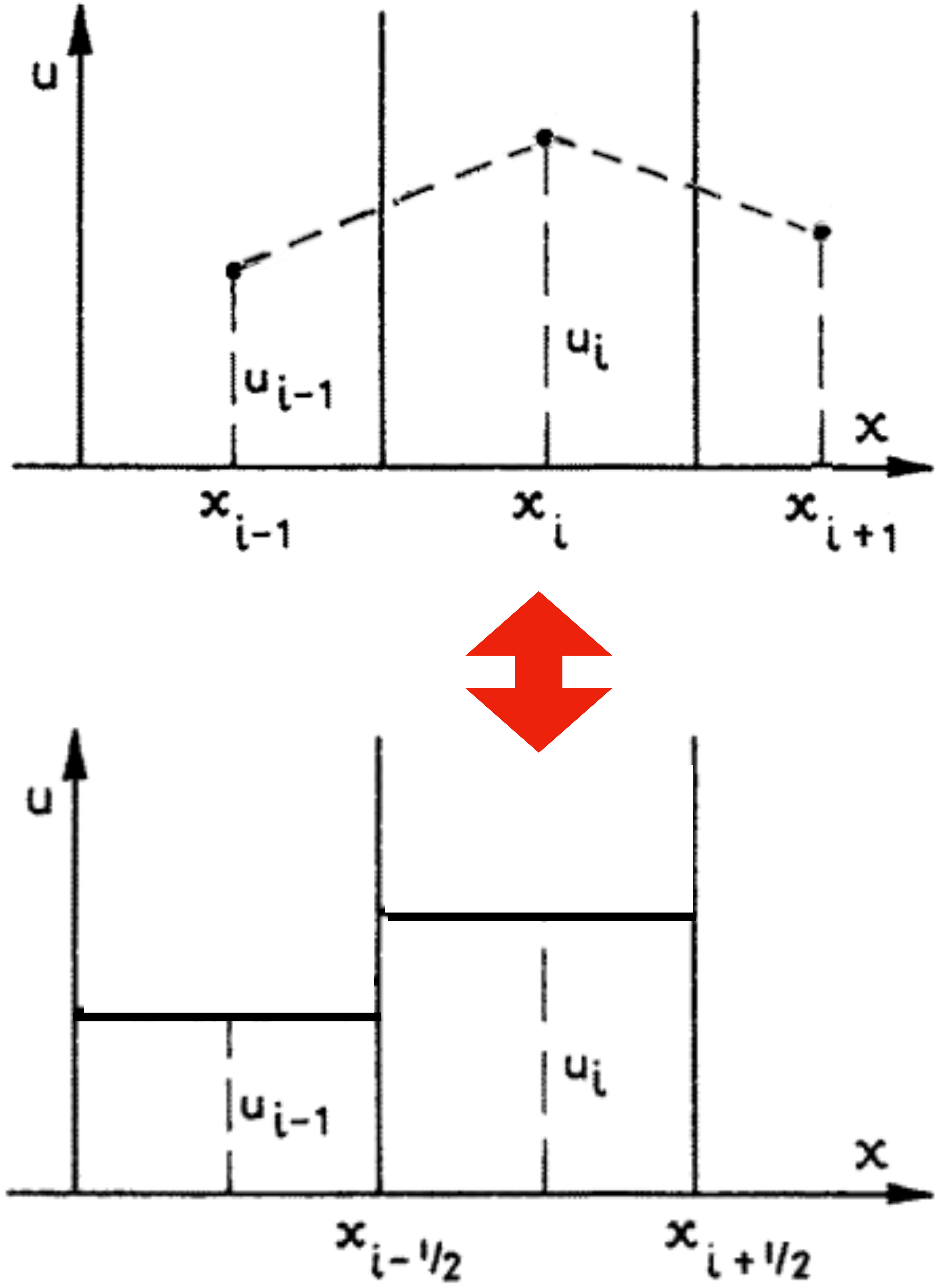
$$\partial_t u + \partial_x f(u) = 0$$

Consistent global fluxes: uFEM/SUPG

$$\hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

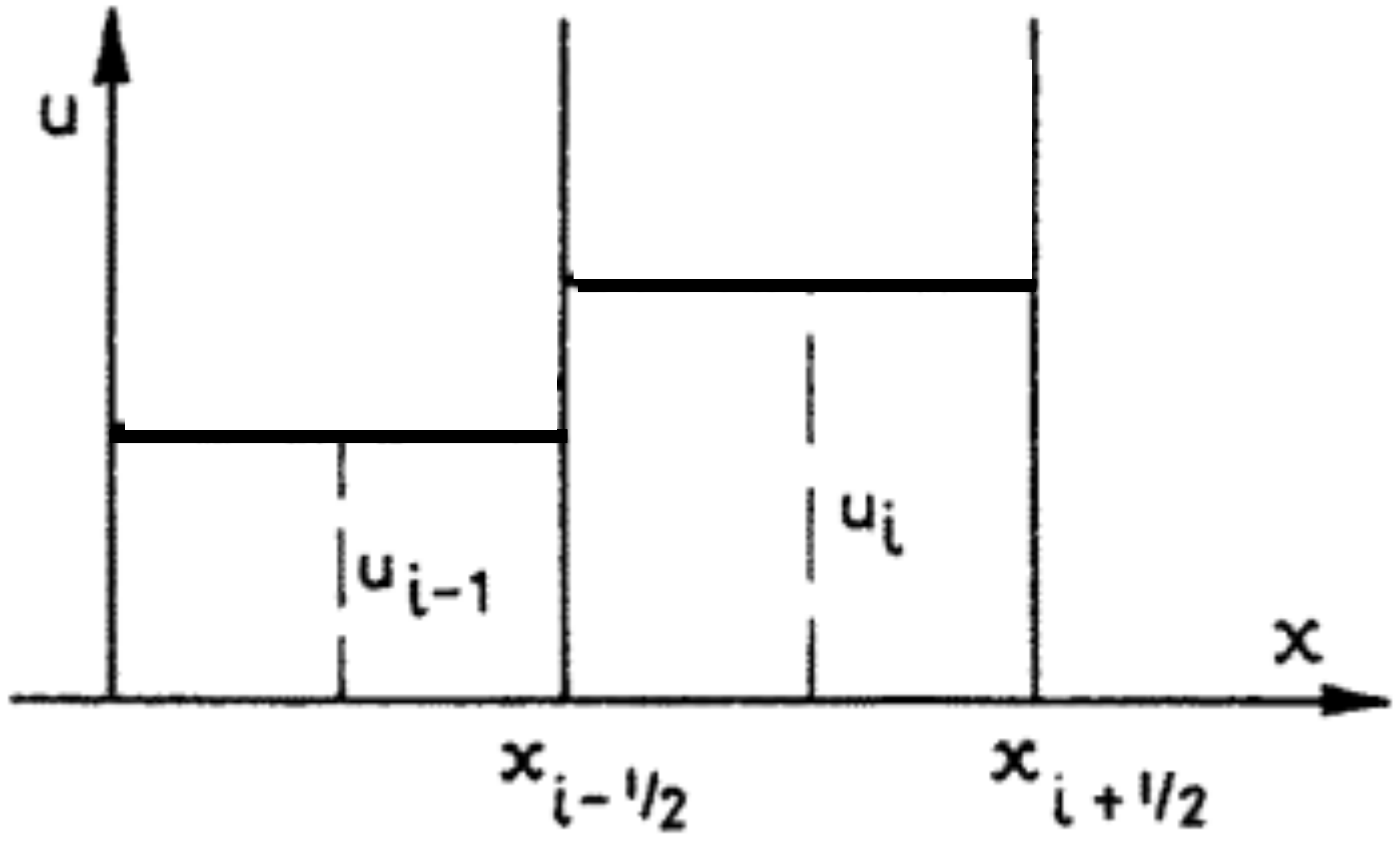
$$\hat{g}_{i+1/2} = g_i + \phi_i^{i+1/2} = g_{i+1} - \phi_{i+1}^{i+1/2}$$

$$= \hat{f}^u_{i+1/2} + \hat{s}_{i+1/2}(\partial_t u)$$



(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$



Consistent global fluxes: uFEM/SUPG

$$\Delta_i x \frac{d\hat{u}_i^{uFEM}}{dt} + \hat{f}^u_{i+1/2} - \hat{f}^u_{i-1/2} = 0$$

$$\Delta_i x \frac{d\hat{u}_i^{uFEM}}{dt} := \Delta_i x \frac{du_i}{dt} + \left\{ \frac{\Delta x}{2} \left(\frac{1}{3} \Delta \frac{du}{dt} + \text{sign}(A) \frac{du}{dt} \right) \right\}_{i+1/2} - \left\{ \frac{\Delta x}{2} \left(\frac{1}{3} \Delta \frac{du}{dt} + \text{sign}(A) \frac{du}{dt} \right) \right\}_{i-1/2}$$

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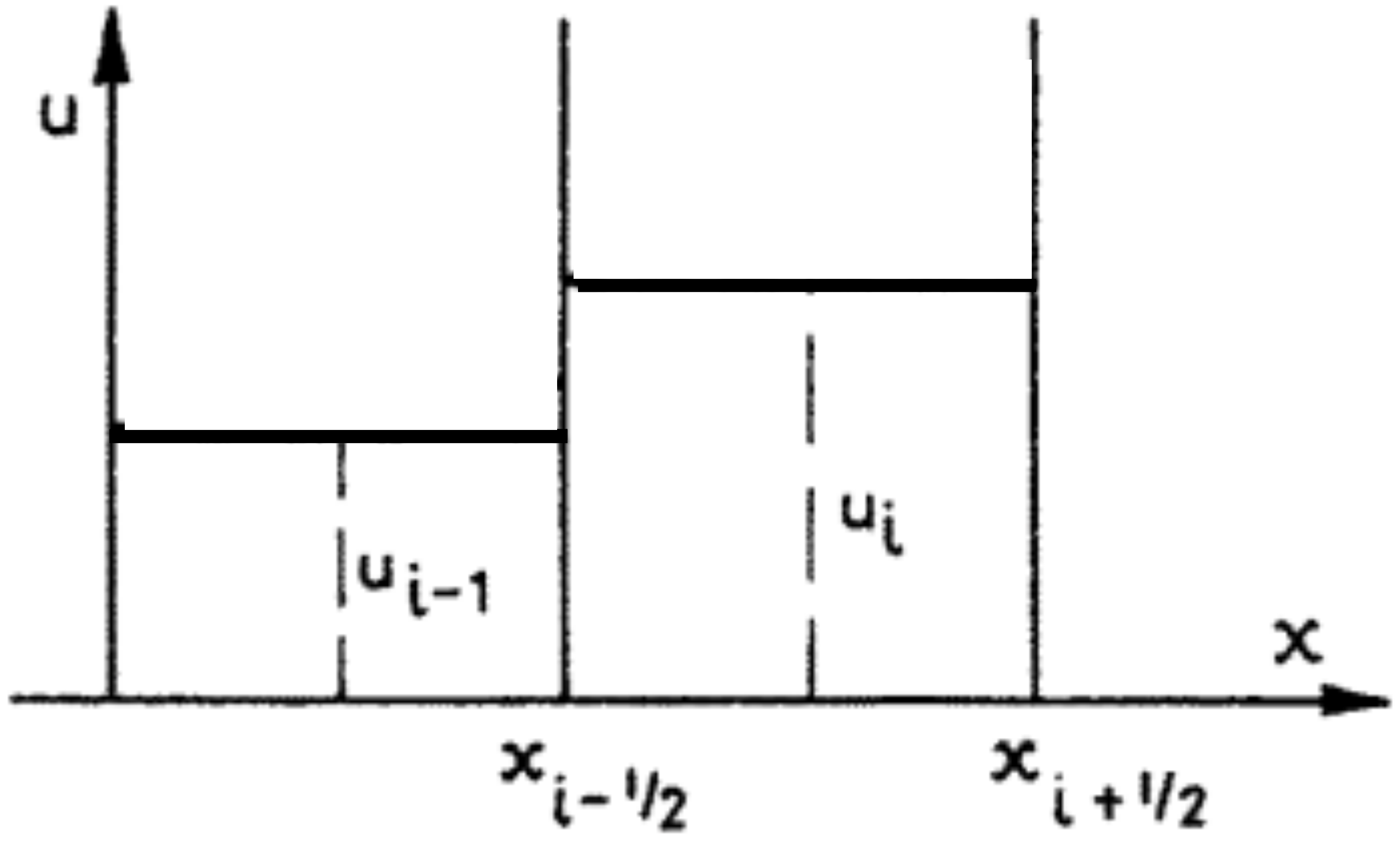
(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$



Consistent global fluxes: uFEM/SUPG

$$\Delta_i x \frac{d\hat{u}_i^{u\text{FEM}}}{dt} + \hat{f}^u_{i+1/2} - \hat{f}^u_{i-1/2} = 0$$

Formal order of accuracy Δx^3 (e.g. by truncated Taylor series)

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Time stepping with mass matrix

$$\Delta_i x \frac{d\hat{u}_i}{dt} + \hat{f}_{i+1/2} - \hat{f}_{i-1/2} = 0$$

$$\Delta_i x \frac{d\hat{u}_i}{dt} = \sum_j m_{ij} \frac{du_j}{dt}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Time stepping with mass matrix: option 1 is go implicit

$$\Delta_i x \frac{d\hat{u}_i}{dt} + \hat{f}_{i+1/2} - \hat{f}_{i-1/2} = 0$$

$$\sum_j m_{ij} \frac{u_j^{n+1} - u_j^n}{\Delta t} + \sum_{l \geq 0} \alpha_l \Delta_i \hat{f}^{n+1-l} = 0\,, \qquad \boxed{\Delta_i \hat{f} := \hat{f}_{i+1/2} - \hat{f}_{i-1/2}}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC* (O2)

$$\Delta_i x \frac{d\hat{u}_i}{dt} + \Delta_i \hat{f} = 0$$

*cf. talks by P Öffner and D. Torlo

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(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x \, \Delta^* u_i + \Delta t \, \Delta_i \hat{f}^n = 0$$

$$\Delta^* u_i := u_i^* - u_i^n \longrightarrow u_i^*$$

$$\Delta_i \hat{f} := \hat{f}_{i+1/2} - \hat{f}_{i-1/2}$$

Euler explicit

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$L^1(u^{n+1}; u^n)$

$$\Delta_i x\, \Delta^* u_i + \Delta t\, \Delta_i \hat{f}^n = 0$$

$L^1(u^*; u^n)$

$$\Delta_i x\, \Delta^{n+1} u_i + \frac{\Delta t}{2}\Delta_i \hat{f}^n + \frac{\Delta t}{2}\Delta_i \hat{f}^* = \Delta_i x\, \Delta^* u_i + \frac{\Delta t}{2}\Delta_i \hat{f}^n + \frac{\Delta t}{2}\Delta_i \hat{f}^*$$
$$- \left\{ \Delta x_i \Delta^* \hat{u}_i + \frac{\Delta t}{2}\Delta_i \hat{f}^n + \frac{\Delta t}{2}\Delta_i \hat{f}^* \right\}$$

$L^2(u^*; u^n)$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x \, \Delta^* u_i + \Delta t \, \Delta_i \hat{f}^n = 0$$

the time derivative is in here

$$\Delta_i x \, \Delta^{n+1} u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{f} = \left\{ \Delta x_i \, \Delta^* u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{f} \right\} - \boxed{\int_{t^n}^{t^{n+1}} \Delta_i \hat{g}}$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x \, \Delta^* u_i + \Delta t \, \Delta_i \hat{f}^n = 0$$

this is what's in the code

$$u_i^{n+1} = u_i^* - \frac{\Delta t}{2}\Delta_i \hat{f}^n - \frac{\Delta t}{2}\Delta_i \hat{f}^* \boxed{- \Delta_i x \, \Delta^* \hat{u}_i}$$

extra correction

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x \, \Delta^* u_i + \Delta t \, \Delta_i \hat{f}^n = 0$$

$$\Delta_i x \, \Delta^{n+1} u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{f} = \left\{ \Delta x_i \, \Delta^* u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{f} \right\} - \int_{t^n}^{t^{n+1}} \Delta_i \hat{g}$$

Higher than second order and ADER variants are possible

(cf. talks by D. Torlo and P Öffner)

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Prototype (notation reminder mostly)

$$\Delta_i x \frac{d\hat{u}_i}{dt} = -\Delta_i \hat{g} = -(\phi_i^{i+1/2} + \phi_i^{i-1/2})$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) + S(u; d) = 0$$

Prototype (notation reminder mostly)

the source term is in here

$$\Delta_i x \frac{d\hat{u}_i}{dt} = \boxed{-\Delta_i \hat{g} = -(\phi_i^{i+1/2} + \phi_i^{i-1/2})}$$

nodal global fluxes are never explicitly evaluated

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \int_{\Delta_{i+1/2}x} (\partial_x f + S)$$

(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x\, \Delta^* u_i = -\Delta t\, \Delta_i \hat{g}^n = -(\phi_i^{i+1/2} + \phi_i^{i-1/2})^n$$

$$\Delta_i x\, \Delta^{n+1} u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{g} = \left\{ \Delta x_i\, \Delta^* u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{g} \right\} - \int_{t^n}^{t^{n+1}} \Delta_i \hat{\mathcal{G}}$$

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(hyperbolic) balance laws

$$\partial_t u + \partial_x f(u) = 0$$

Fully explicit schemes: predictor-corrector/DeC (O2)

$$\Delta_i x\, \Delta^* u_i = \boxed{-\Delta t\, \Delta_i \hat{g}^n = -\,(\phi_i^{i+1/2} + \phi_i^{i-1/2})^n}$$ the source term is in here

$$\Delta_i x\, \Delta^{n+1} u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{g} = \left\{ \Delta x_i\, \Delta^* u_i + \int_{t^n}^{t^{n+1}} \Delta_i \hat{g} \right\} - \boxed{\int_{t^n}^{t^{n+1}} \Delta_i \hat{\mathscr{G}}}$$

source term and time derivative are in here

1D examples using RD

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)

$$\partial_t \mathrm{u} + \partial_x \mathrm{f}(\mathrm{u}) + \mathrm{S}(\mathrm{u};\mathrm{d}) = 0$$

⇩

$$\partial_t f^\epsilon + \Lambda \partial_x f^\epsilon + \tilde{\mathrm{s}}^\epsilon + \frac{f^\epsilon - \mathrm{M}(\mathrm{P}f^\epsilon)}{\epsilon} = 0$$

Standard discrete kinetic approach

Jin and Xin Comm. Pure Appl. Math. 1995

D. Aregba-Driollet and R. Natalini, SINUM 2000

etc.

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)

$$\partial_t \mathrm{u} + \partial_x \mathrm{f}(\mathrm{u}) + \mathrm{S}(\mathrm{u}; \mathrm{d}) = 0$$

$$\Downarrow$$

$$\partial_t f^\epsilon + \Lambda \partial_x f^\epsilon + \tilde{\mathrm{s}}^\epsilon + \frac{f^\epsilon - \mathrm{M}(\mathrm{P} f^\epsilon)}{\epsilon} = 0$$

Perturbation to account for bathymetry and friction: $\mathrm{P}\tilde{\mathrm{s}}^\epsilon = \mathrm{S}(\mathrm{P}f^\epsilon; \mathrm{d})$

Delis and Katsaounis, IJNMF 2003

Delis and Katsaounis, Appl.Math.Mod. 2005

etc.

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)

$$\boxed{f_i^{\epsilon,n+1} - f_i^{\epsilon,*} + \frac{\Delta t}{\epsilon}\left(f^{\epsilon,n+1} - M^{n+1}\right)_i} + \frac{\Delta t}{\Delta_i x} \sum_{C \ni i} \boxed{\phi_i(f^{\epsilon,*}, f^{\epsilon,n})} = 0$$

IMEX part

RD + DeC:

time derivative, sources, relaxation terms

Discretization by well balanced RD + IMEX DeC

Abgrall and Torlo, SISC 2020

MR, J.Comput.Phys. 2015

etc.

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)

Discretization by well balanced RD + IMEX DeC

Formal equivalence (CE expansion, within ϵ) with DK formulation of global flux PDE

$$\partial_t \mathrm{u}^\epsilon + \partial_x g(\mathrm{u}^\epsilon; \mathrm{d}) = \mathcal{O}(\epsilon)$$

The remainder is non-vanishing in both formulations, and can be written as

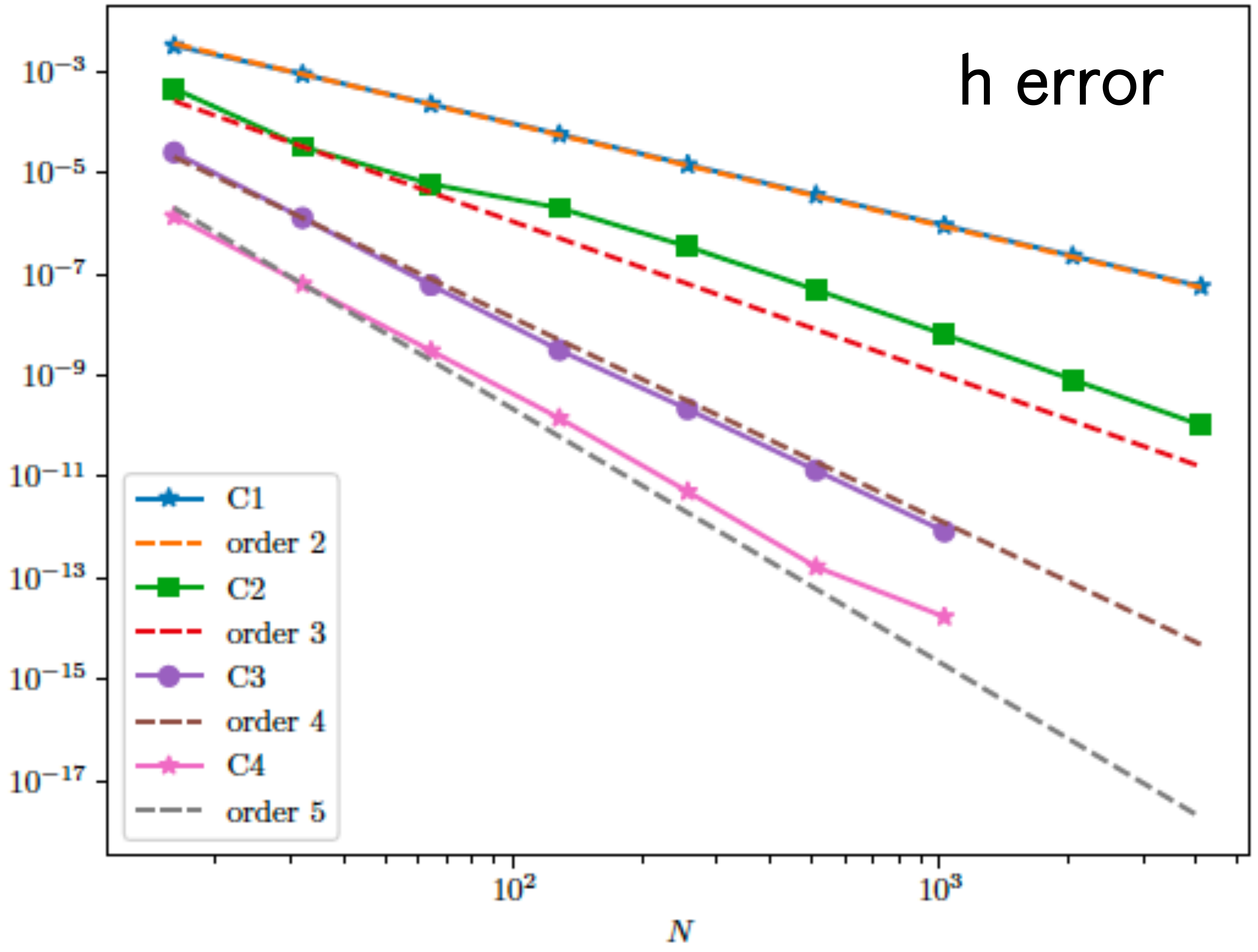
$$\epsilon \partial_x (\mathrm{B}^\epsilon \partial_x \mathrm{u}^\epsilon) + \epsilon \partial_x \Delta_S$$

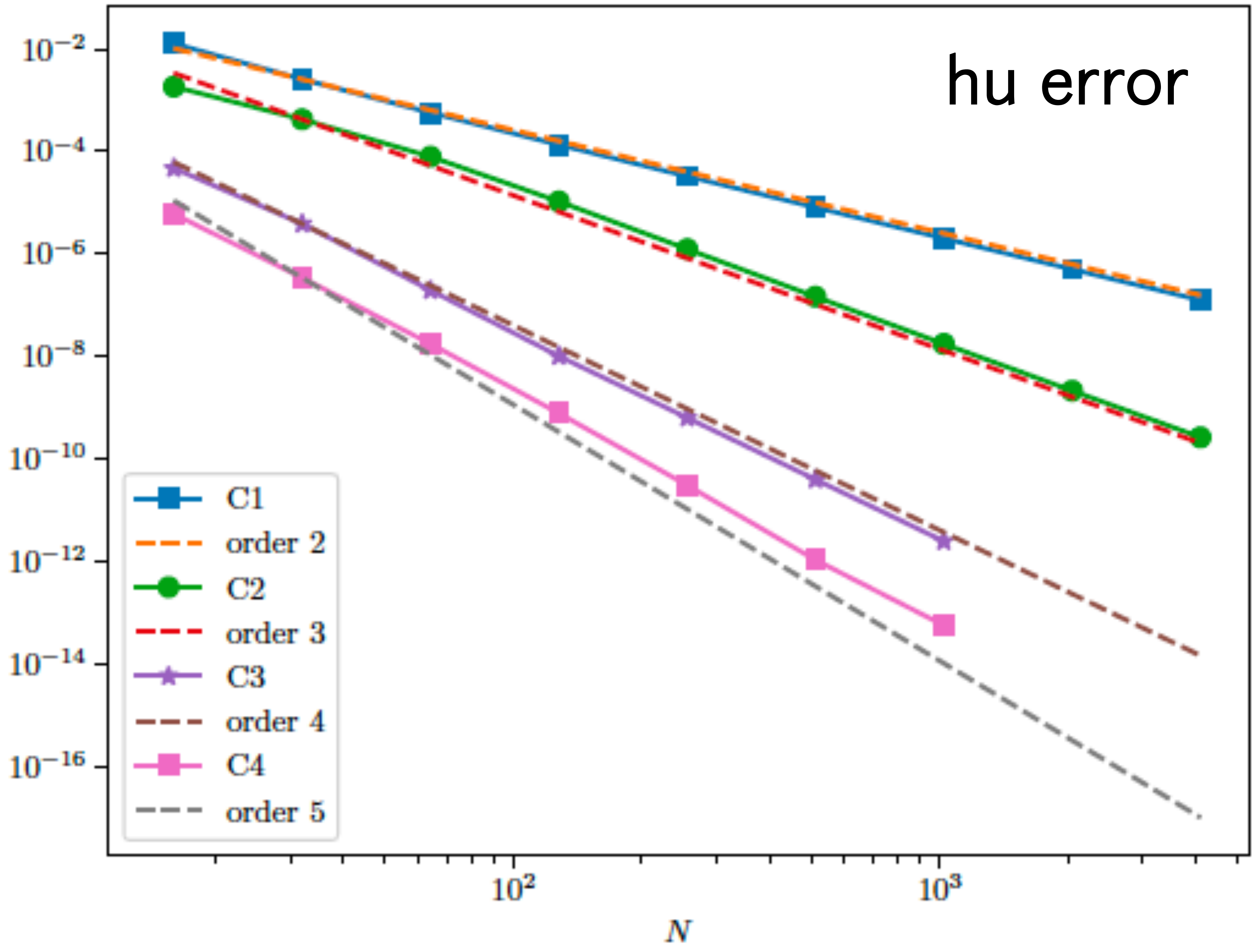
Spatial/temporal derivatives of the source(flux)

Positive definite under a sub-char. condition

Discrete Kinetic IMEX-RD for shallow water

Jointly with D. Torlo (CARDAMOM Inria Bordeaux Sud-Ouest, France)





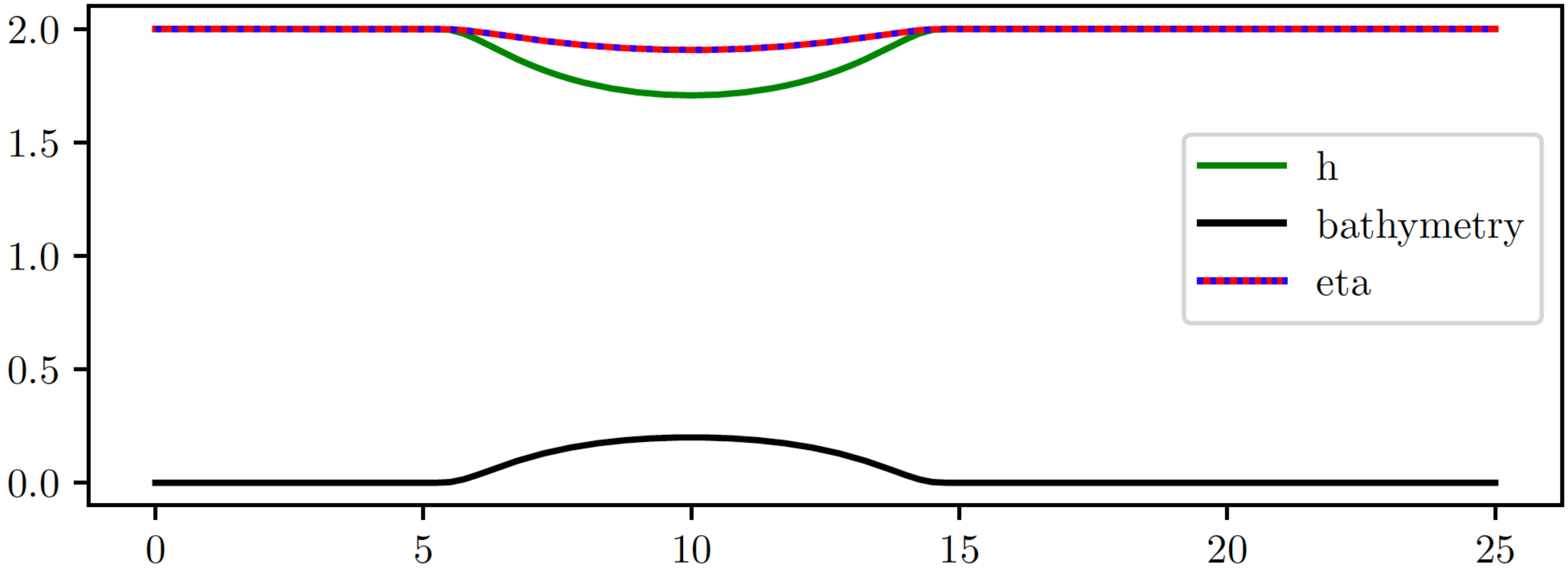
Manufactured (Steady) smooth solution:

kinetic var.s initialised using Maxwellians, and computation for finite times

(linear schemes)

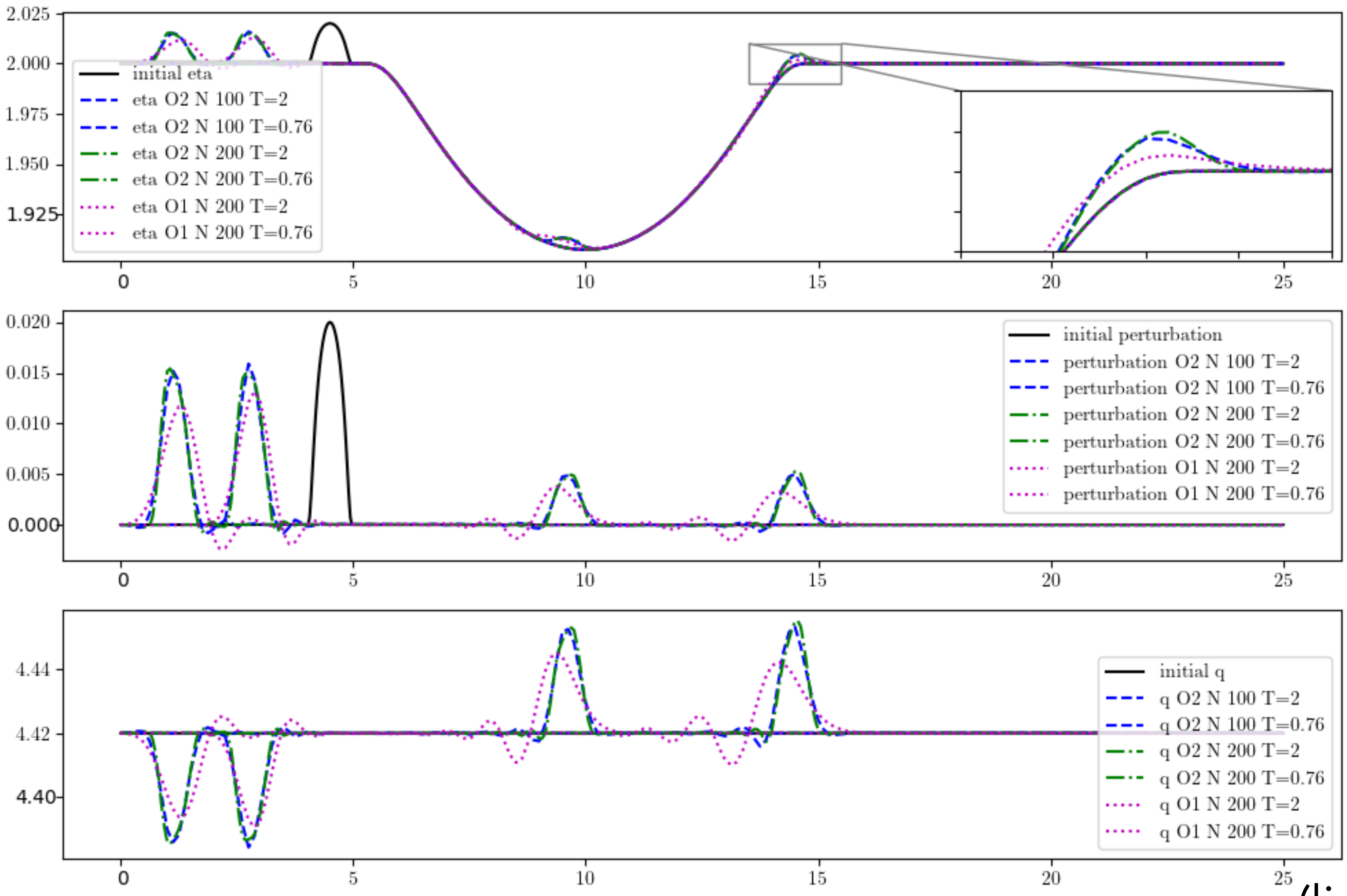
Discrete Kinetic IMEX-RD for shallow water

Perturbation of the classical subcritical case (linear schemes)



Delestre et al, IJNMF 2013

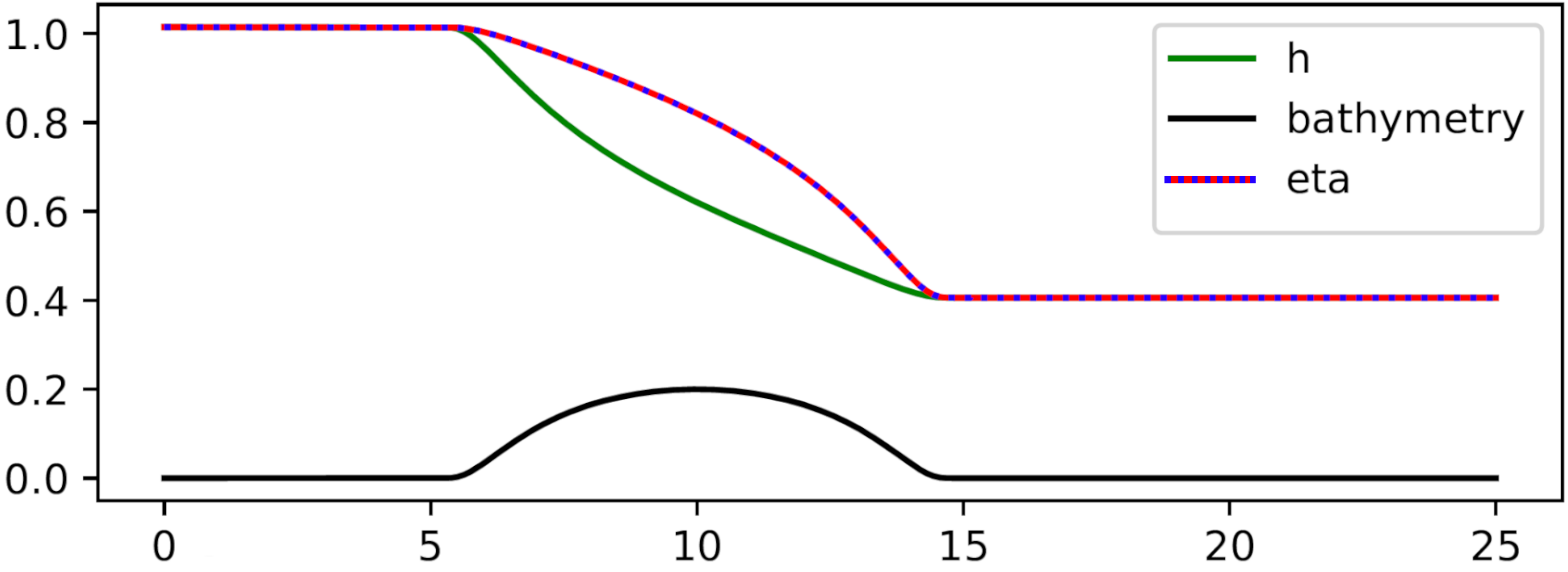
Discrete Kinetic IMEX-RD for shallow water



(linear schemes)

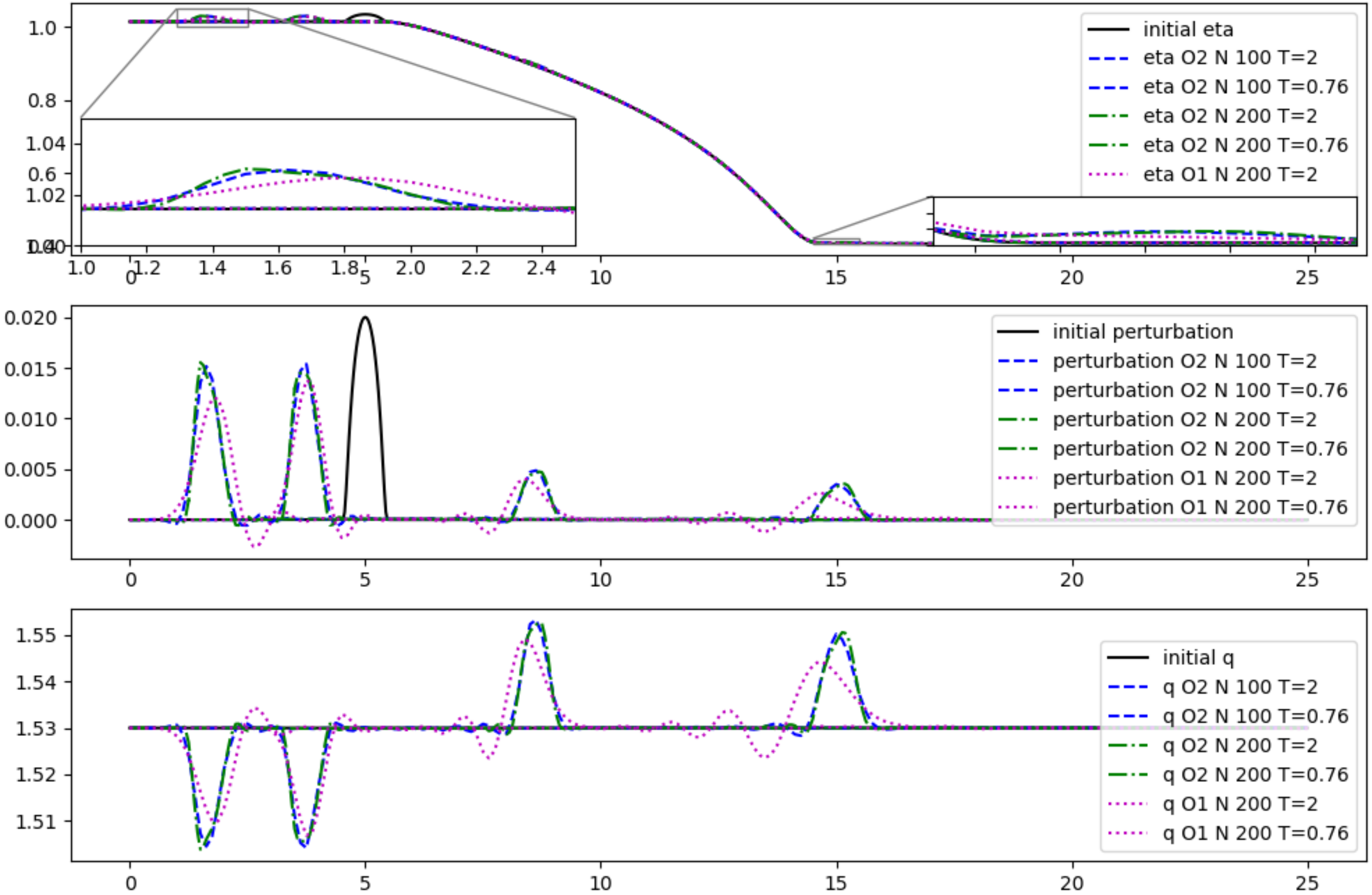
Discrete Kinetic IMEX-RD for shallow water

Perturbation of the classical transcritical case (linear schemes)



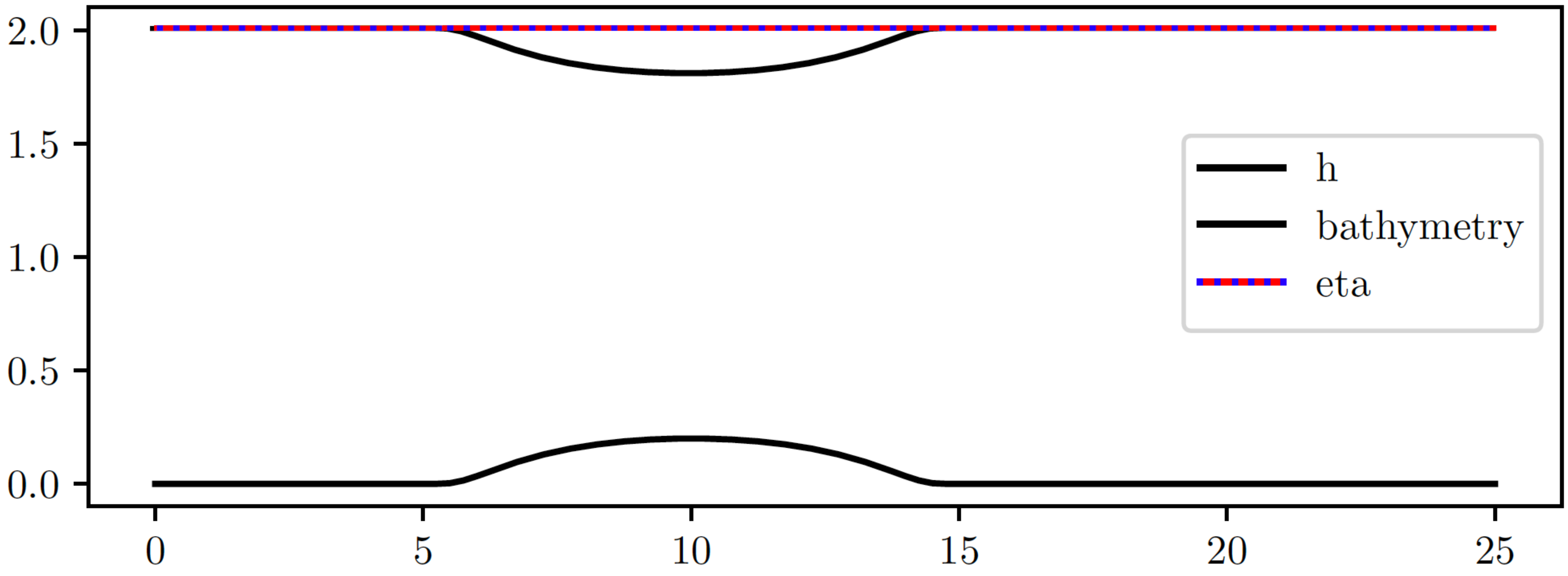
Delestre et al, IJNMF 2013

Discrete Kinetic IMEX-RD for shallow water



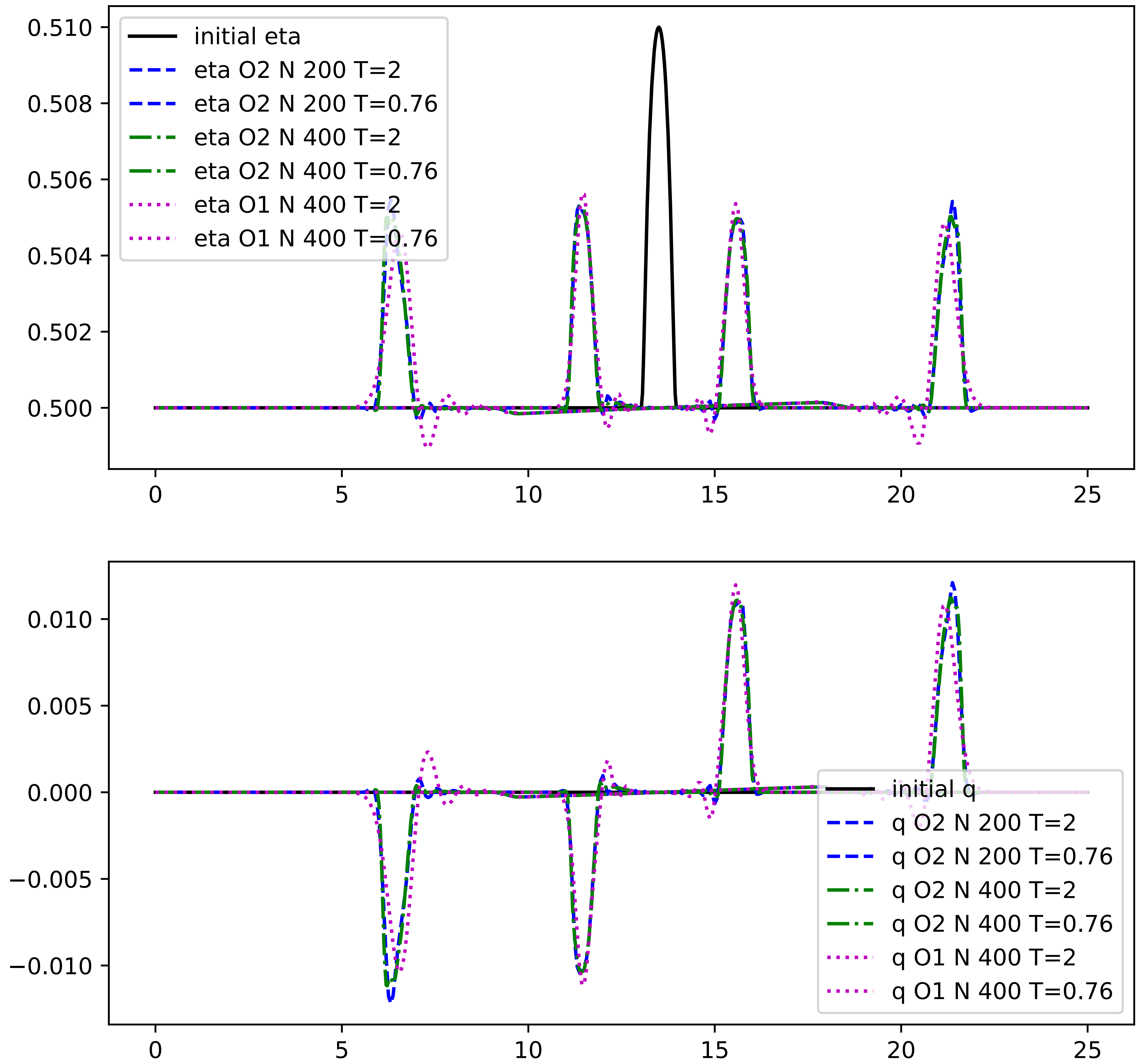
Discrete Kinetic IMEX-RD for shallow water

Perturbation of the classical lake at rest case (linear schemes)



Delestre et al, IJNMF 2013

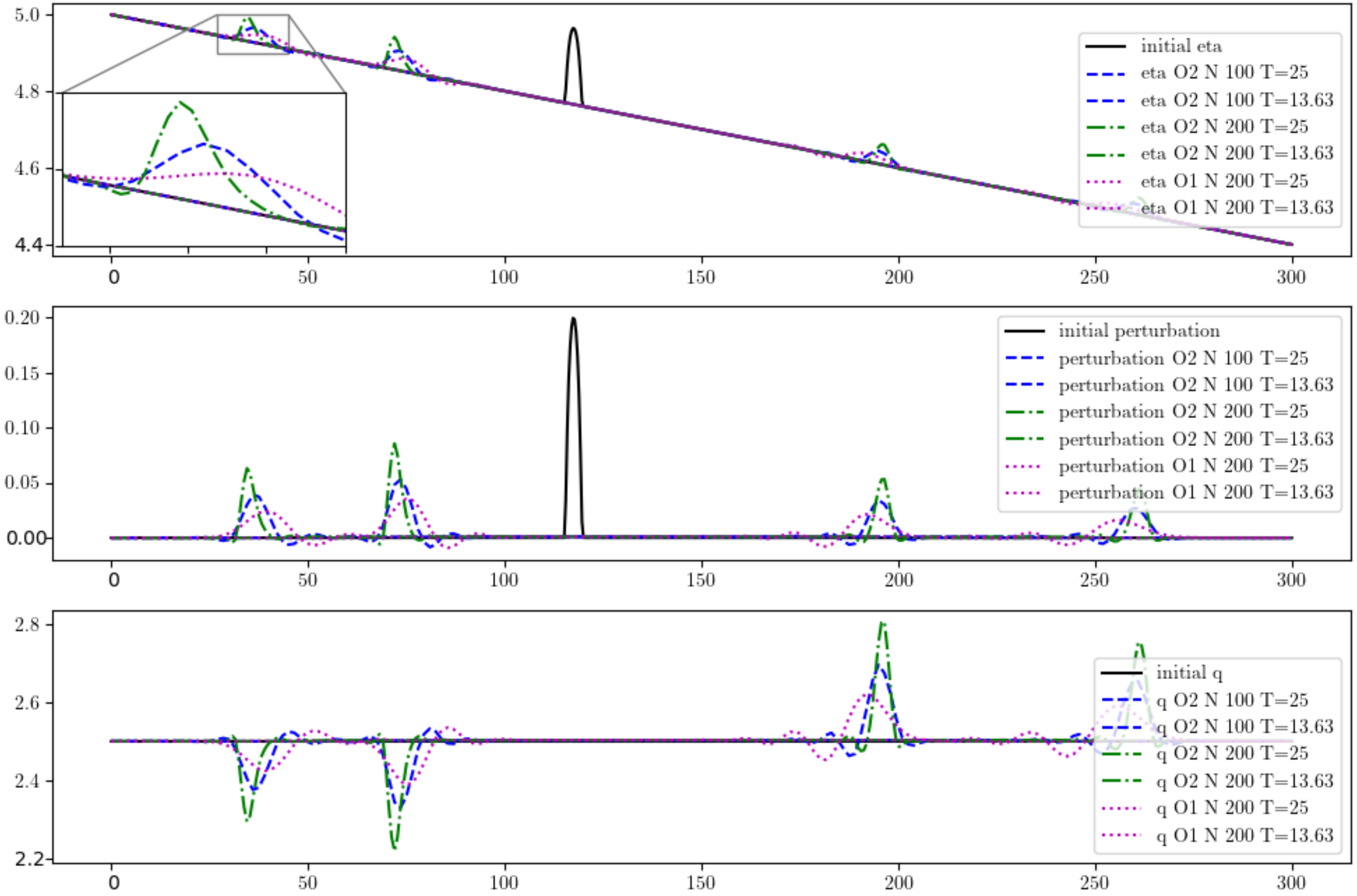
Dispersive shallow water waves



(linear schemes)

Dispersive shallow water waves

Perturbation of the constant slope equilibrium : $gh\nabla b = -c_f(h, \|\vec{u}\|)\vec{u}$



(linear schemes)

Dispersive shallow water waves

Dispersive shallow water waves

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathscr{D}$$

$$\mathscr{D} = Bd^2\partial_{xxt}(hu) + \frac{d\partial_x d}{3}\partial_{xt}(hu) + \beta g d^3\partial_{xxx}\eta - 2\beta g d^2\partial_x d\partial_{xx}\eta$$

depth at rest free surface

$$d = h_0 - b\ ,\quad \eta = h + b$$

Dispersive shallow water waves

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathscr{D}$$

conservative flux,
topography, dispersion

$$\sum_j m_{ij} \frac{U_j^{n+1} - U_j^n}{\Delta t} + \sum_{l \geq 0} \alpha_l \sum_{C \ni i} \boxed{\phi_i(U^{n+1-l})} = 0$$

Discretization by well balanced RD + implicit time stepping

MR and A. Filippini, J.Comput.Phys. 2014
R. Abgrall and MR, ECM 2017
etc.

Solitary waves

Solitary waves travelling (on flat bathymetry) at speed C solution of the ODE

$$C^2 h_0^2 (B - \beta \frac{C_0^2}{C^2}) \eta'' + C^2 (\frac{C_0^2}{C^2} - 1)\eta + \frac{g}{2}\eta^2 + C\frac{\eta}{\eta + h_0} = 0$$

with $(h, hu) = (\eta - b, C\eta)$ and $C_0^2 = gh_0$

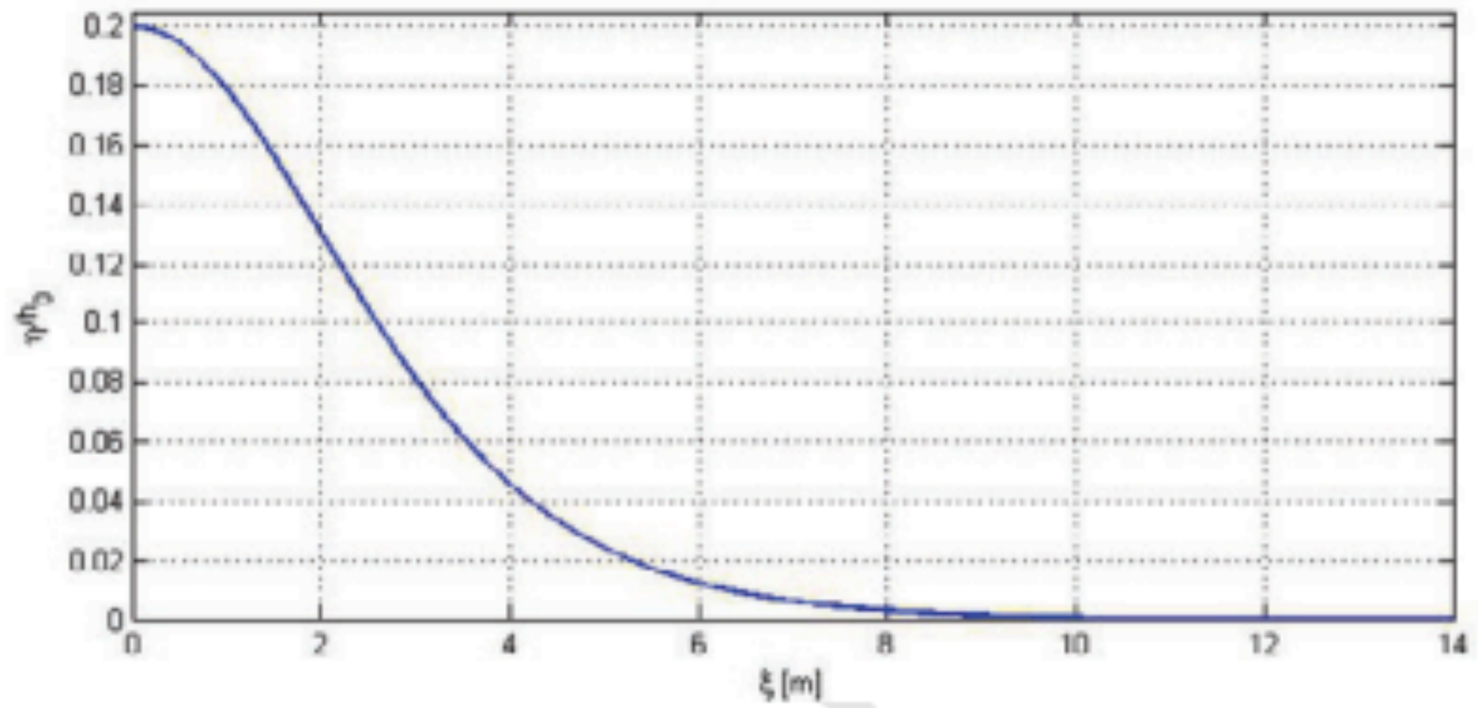
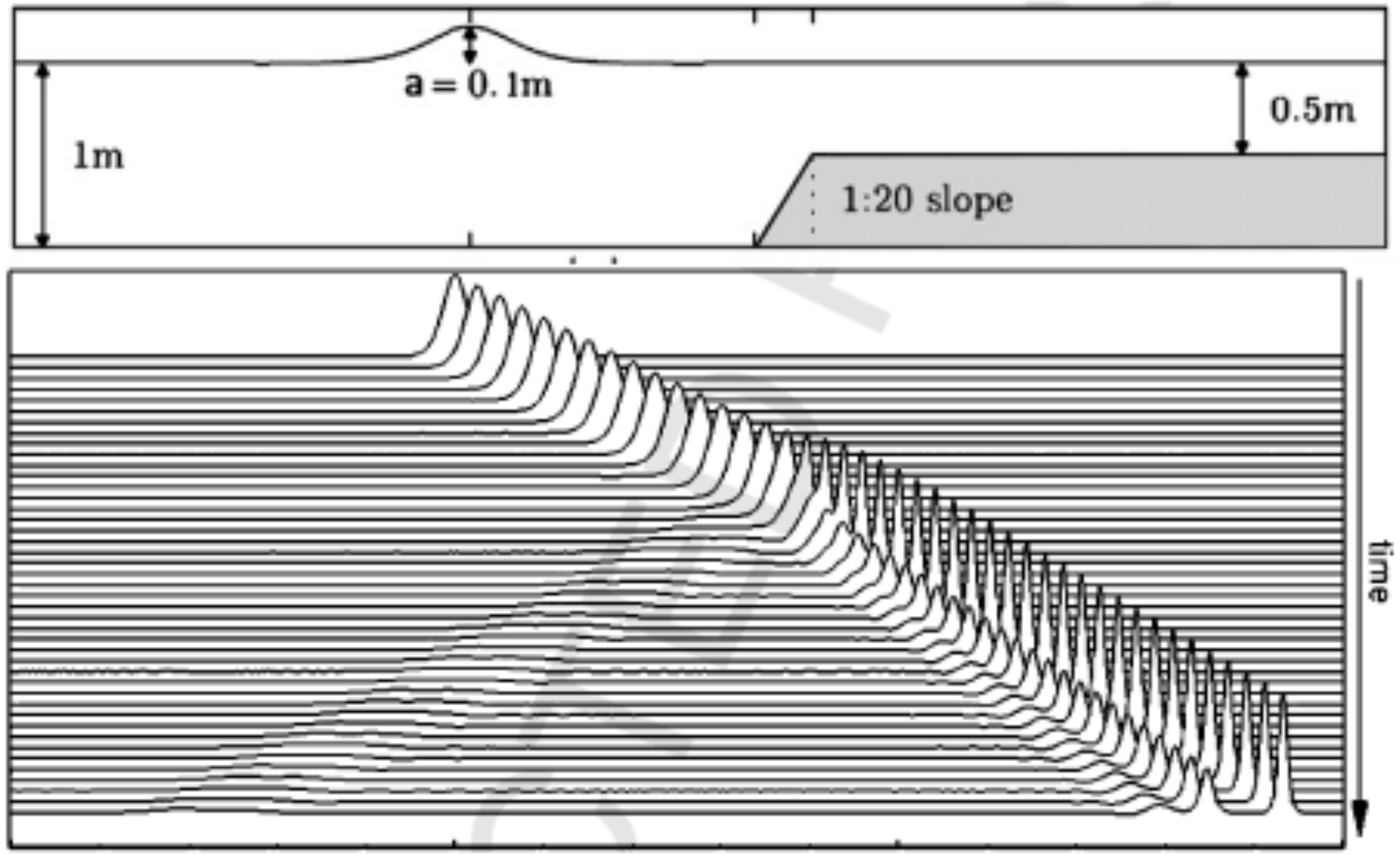


Figure 5: Soliton profile for $h_0 = 1\,[m]$ and $A/h_0 = 0.2$ obtained by numerically integrating (34)

Solitary wave shoaling and fission



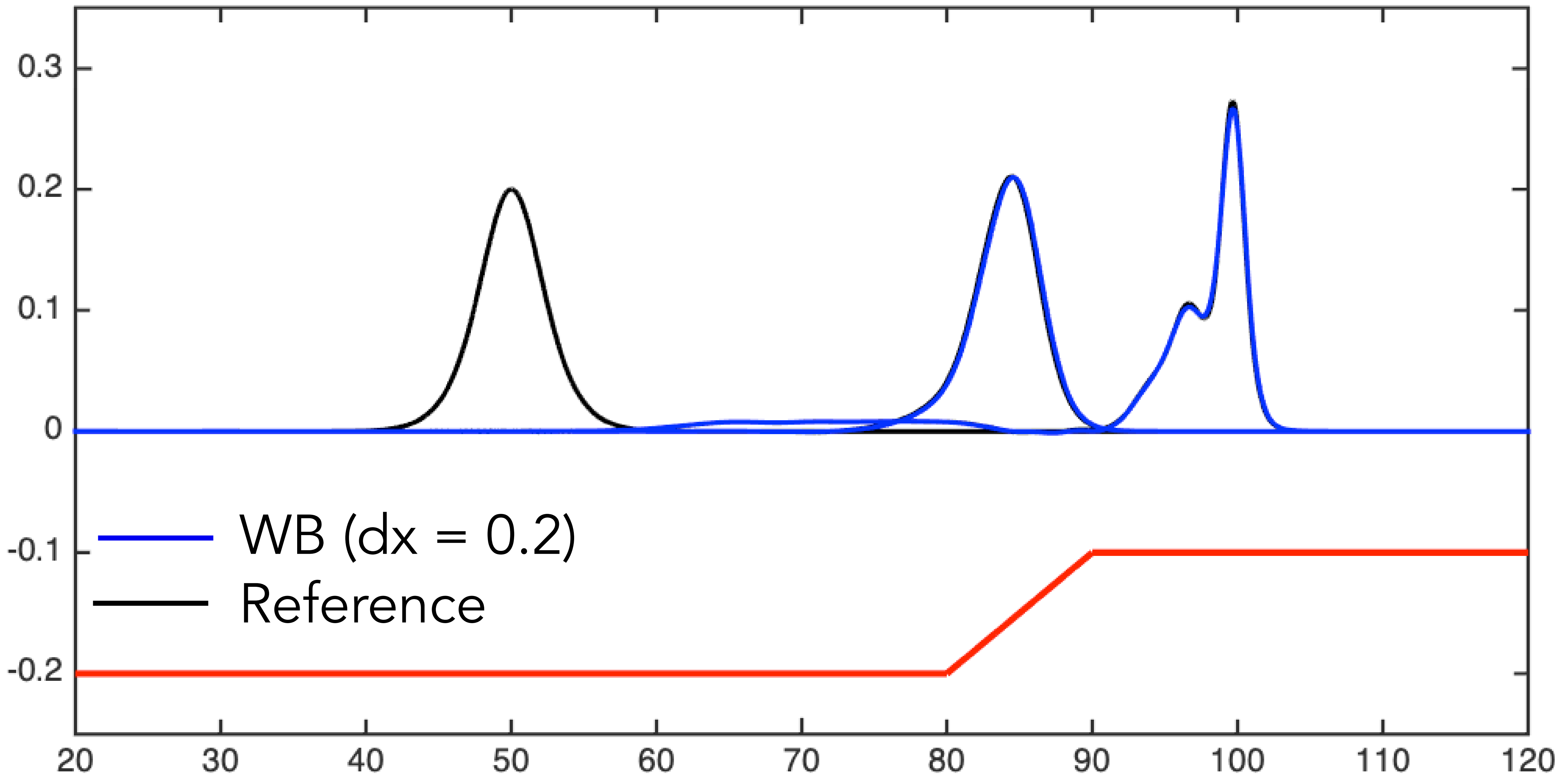
$$\partial_t h + \partial_x(hu) = 0$$

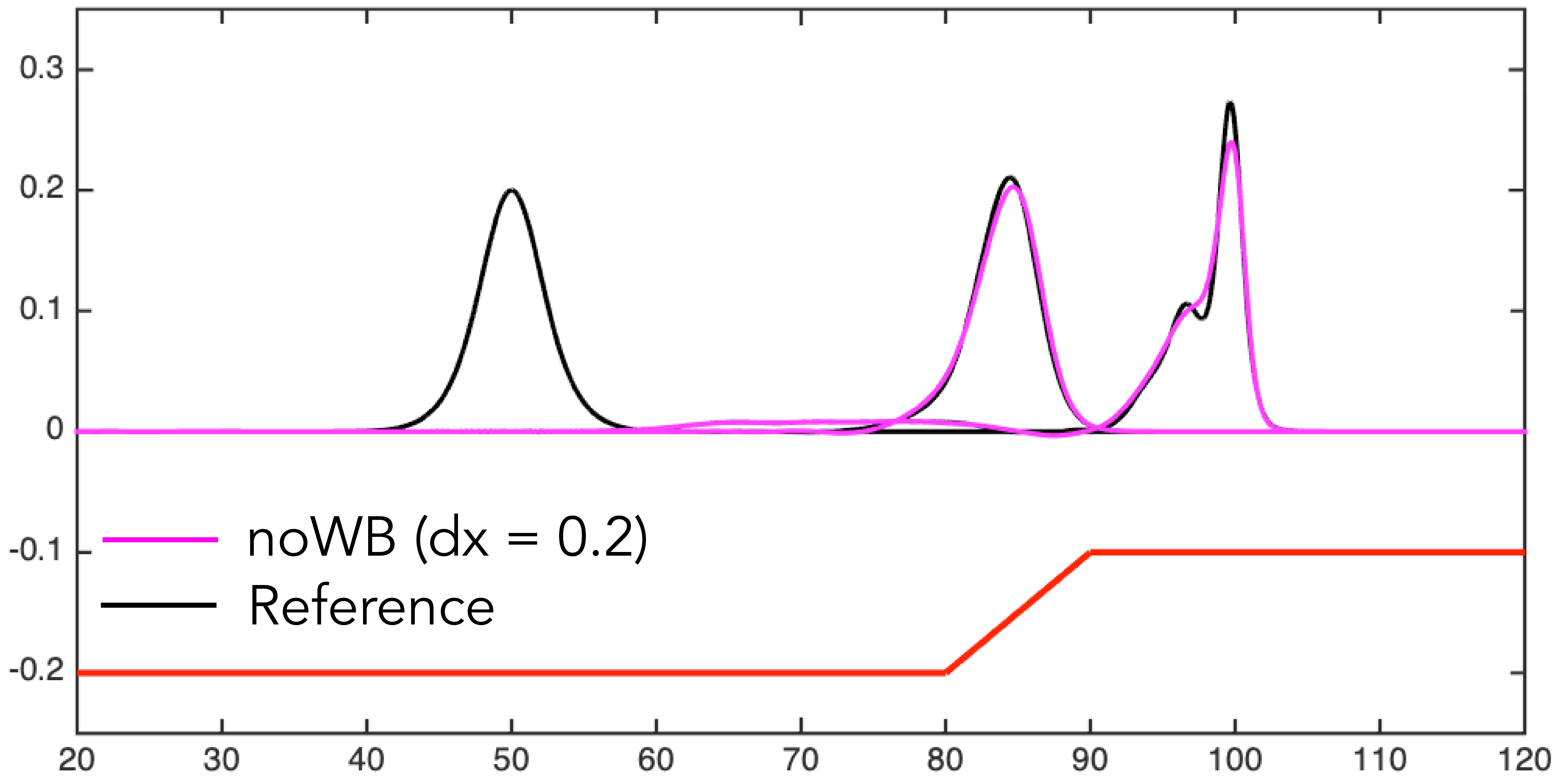
$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) + gh\partial_x b = \mathscr{D}$$

Well Balanced: $\phi_i = \beta_i \int (\partial_x f - \mathscr{D})$

no WB: $\phi_i = \beta_i \int \partial_x f - \int \varphi_i \mathscr{D}$

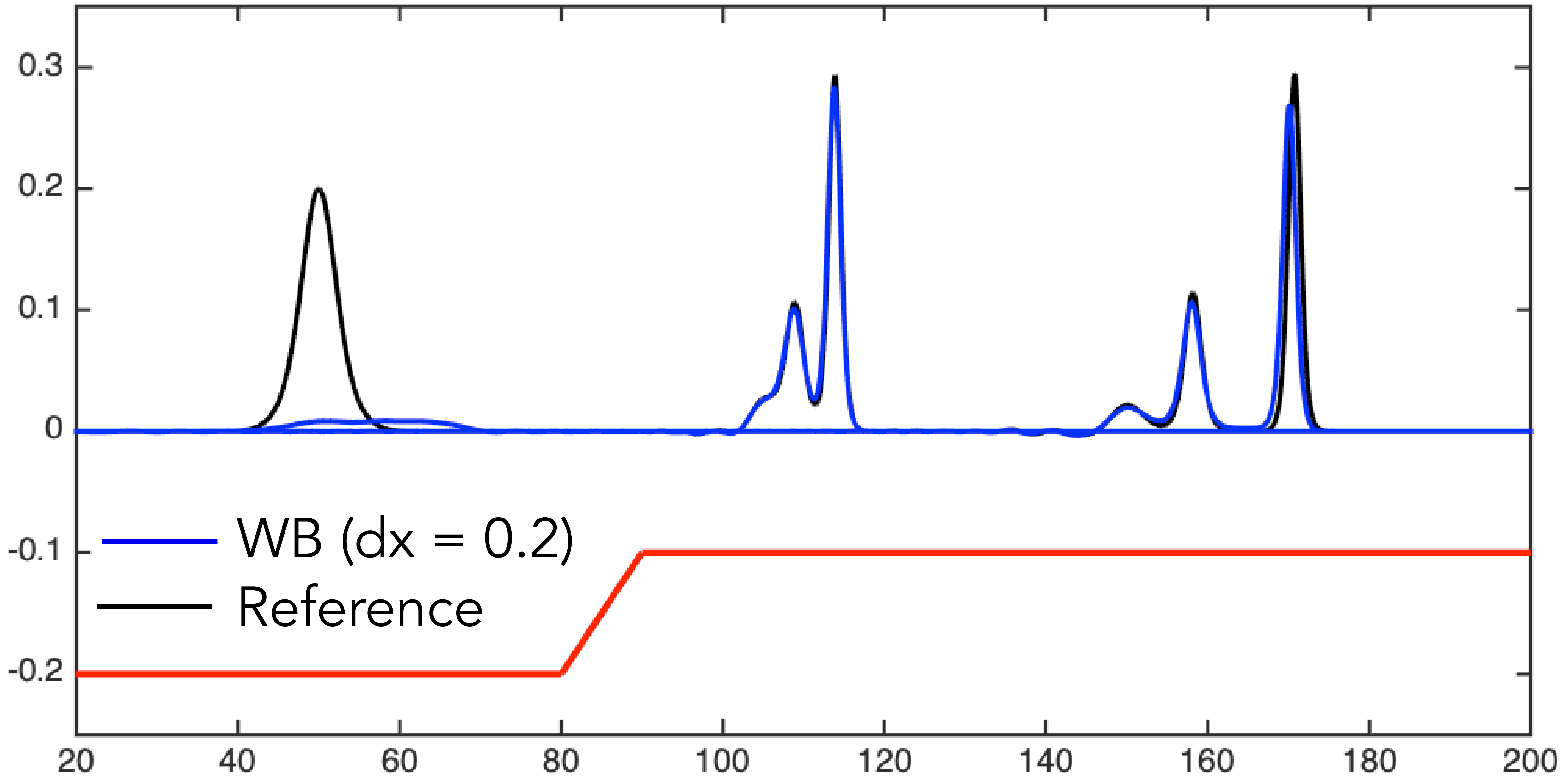
Solitary wave shoaling and fission

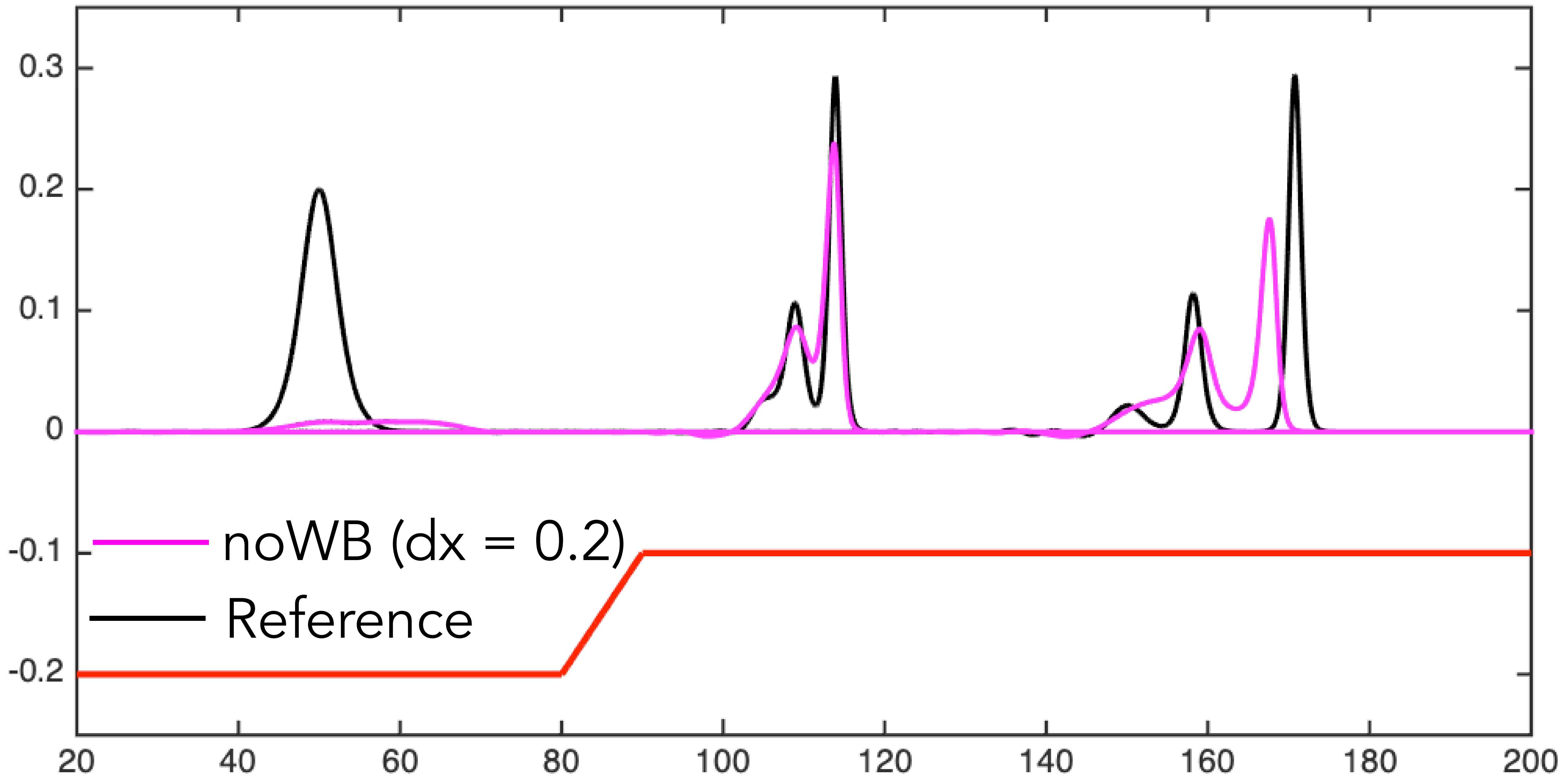




short time

Solitary wave shoaling and fission





long(er) time

Solitary wave shoaling and fission

(hyperbolic) balance laws: 1d summary

Error balance and residual distribution/fluctuation splitting

- conservation laws —> equivalence with FV and consistent fluxes
- balance laws —> equivalence with FV and **consistent global fluxes**
- global flux and unsteady: mass matrix and error balance
- Time stepping: implicit or explicit predictor/corrector aka Defect Correction
- variety of source terms and so far no a-priori knowledge on the ex. sol !

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Consistent global fluxes

Dimension by dimension extension: 1D consistency along on mesh aligned directions or more ? Nice results in e.g. (Chertock et al, JCP 2018)

$$\partial_t u + \partial_x g(u; x) = 0$$

$$g(u; x) = f(u) + s(u; x)$$

$$s(u; x) = \int_{x_0}^{x} S(u(s); d(s))$$

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Consistent global fluxes

Solenoidal involution !

$$\partial_t u + \nabla \cdot G(u; x) = 0$$

$$G(u; x) = f(u) + \sigma(u; x)$$

$$\nabla \cdot \sigma = S$$

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Consistent global fluxes

$$\partial_t u + \nabla \cdot G(u; x) = 0$$

$$G(u; x) = f(u) + \sigma(u; x)$$

$$\nabla \cdot \sigma = S$$

Solenoidal involution !

- staggered methods
- vector decomposition & potentials

Hyman and Shashkov, CAMWA 1997
Mishra and Tadmor, CCP 2011 and SINUM 2011
etc

Closing(1). Going multiD

Back to 1d to change paradigm

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0 \quad \Leftrightarrow \quad \Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} = \int_{\Delta_{i+1/2}x} (\partial_x f + S)$$

$$\hat{g}_{i+1/2} = g_i + \phi_i^{i+1/2} = g_{i+1} - \phi_{i+1}^{i+1/2}$$

$$g_i := f_i + s_i$$

focus on this

$$\boxed{s_i := s_{i-1} + \Delta_{i-1/2}x \, S_{i-1/2}}$$

Closing(1). Going multiD

Back to 1d to change paradigm

$$s_{\mathrm{i}} := s_{\mathrm{i-1}} + \Delta \mathrm{x}\, \mathrm{S}_{\mathrm{i-1/2}} \approx \int_{x_{i-1}}^{x_i} \partial_{\mathrm{x}} s(\mathrm{x}) = \int_{\mathrm{x_{i-1}}}^{\mathrm{x_i}} \mathrm{S}$$

$$\frac{s_{\mathrm{i}} - s_{\mathrm{i-1}}}{\Delta \mathrm{x}} = \mathrm{S}_{\mathrm{i-1/2}} \approx \partial_{\mathrm{x}} s_{\mathrm{i-1/2}} = S_{\mathrm{i-1/2}}$$

cell potential for the source flux

$$\psi_{\mathrm{i+1/2}} := \psi_{\mathrm{i-1/2}} - \Delta x\, s_{\mathrm{i}}$$

$$\boxed{-\frac{\psi_{\mathrm{i+1/2}} - 2\psi_{\mathrm{i-1/2}} + \psi_{\mathrm{i-3/2}}}{\Delta \mathrm{x}^2} = \mathrm{S}_{\mathrm{i-1/2}} \approx -\,\partial_{\mathrm{xx}}\psi_{\mathrm{i-1/2}} = \mathrm{S}_{\mathrm{i-1/2}}}$$

Closing(1). Going multiD

Back to 1d to change paradigm

$$\Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \phi_i^{i-1/2} + \phi_i^{i+1/2} = 0 \quad \Leftrightarrow \quad \Delta_i x \frac{u_i^{n+1} - u_i^n}{\Delta t} + \hat{g}_{i+1/2} - \hat{g}_{i-1/2} = 0$$

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \phi^{i+1/2} = \int_{\Delta_{i+1/2}x} (\partial_x f + S)$$

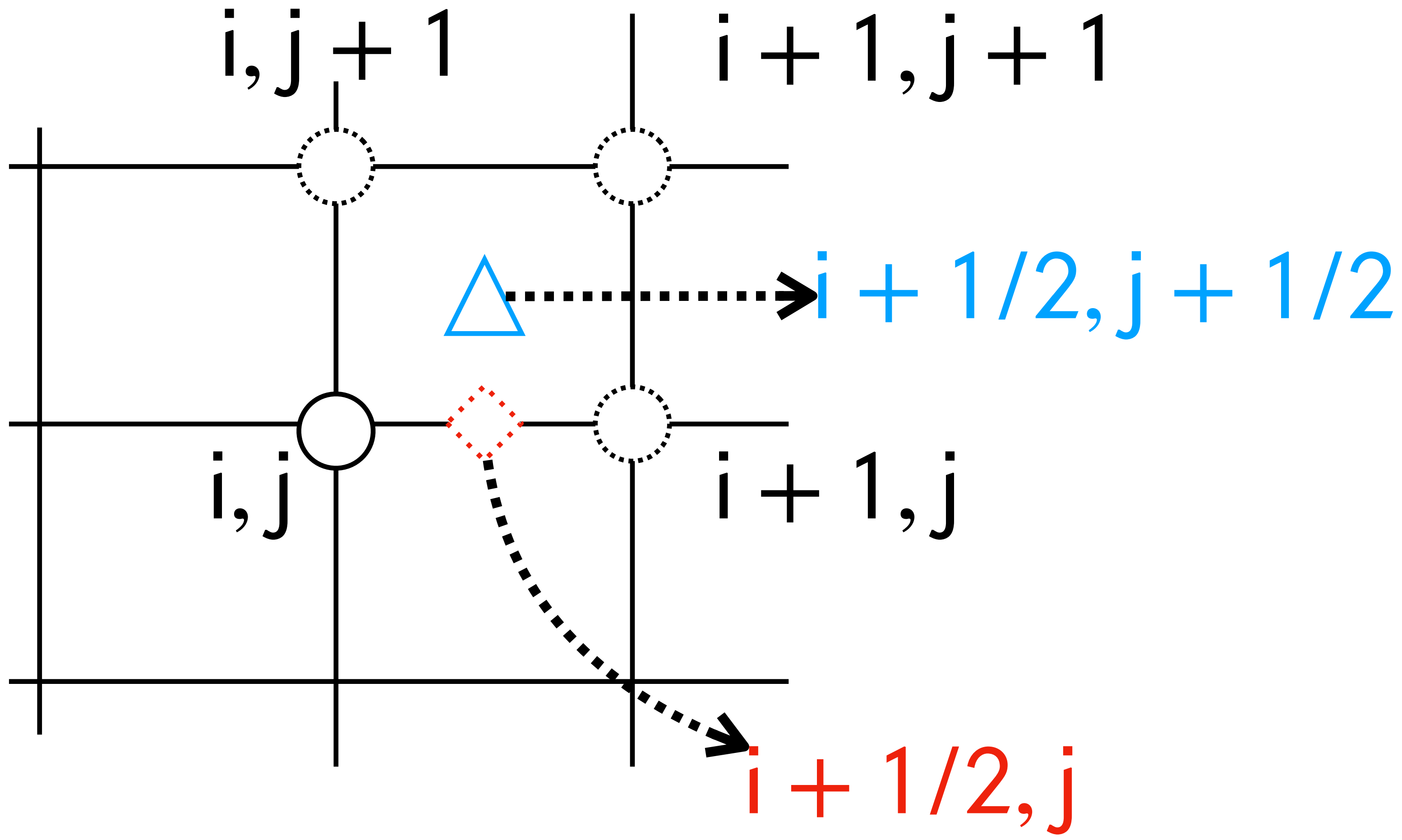
used in practice

$$g_i := f_i - \frac{\psi_{i+1/2} - \psi_{i-1/2}}{\Delta x}$$

$$-\frac{\psi_{i+1/2} - 2\psi_{i-1/2} + \psi_{i-3/2}}{\Delta x^2} = S_{i-1/2}$$

Inria

Closing(1). Going multiD



$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Residual distribution and global fluxes : Cartesian grids

$$|C_{i,j}| \frac{du_{i,j}}{dt} + \sum_{C \ni i,j} \phi^C_{i,j} = 0$$

$$\sum_{i,j \in C} \phi^C_{i,j} = \phi^C = \int_C (\nabla \cdot F + S)$$

For example $\phi^C_{i,j} = \beta_{i,j} \phi^C$

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Residual distribution and global fluxes : Cartesian grids

Retrace the steps taken in 1D

- STEP 1: write equivalent consistent fluxes for RD in homogeneous case

- STEP 2: what is the source integral equal to in 2D ?

Closing(1). Going multiD

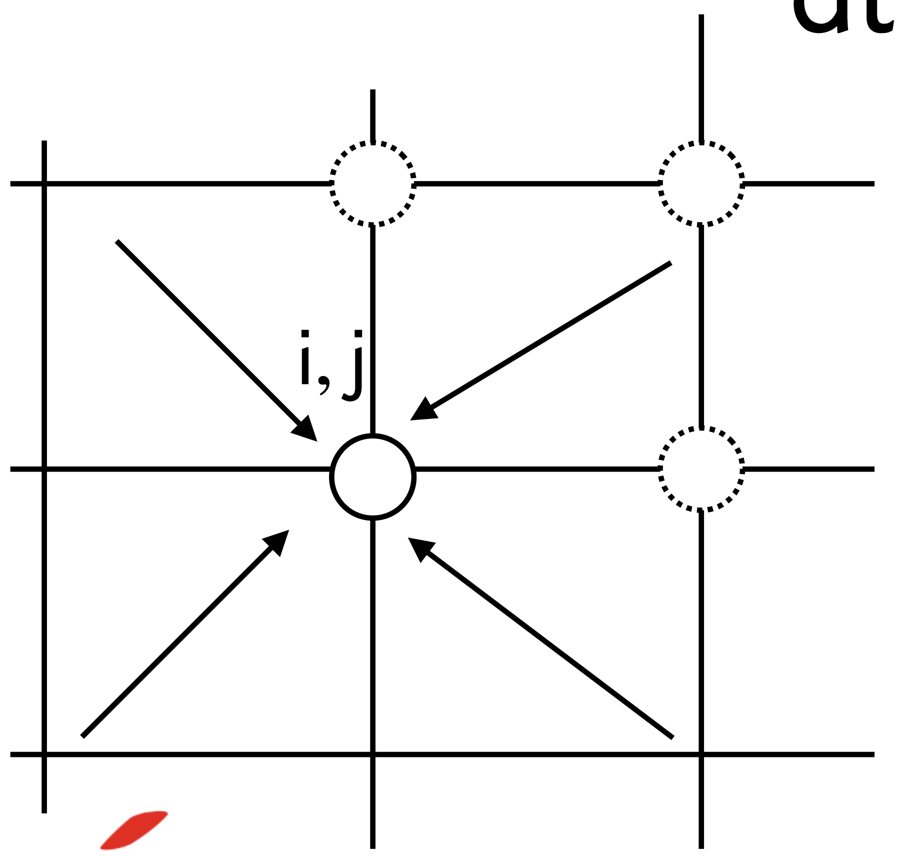
$$\partial_t u + \nabla \cdot F(u) = 0$$

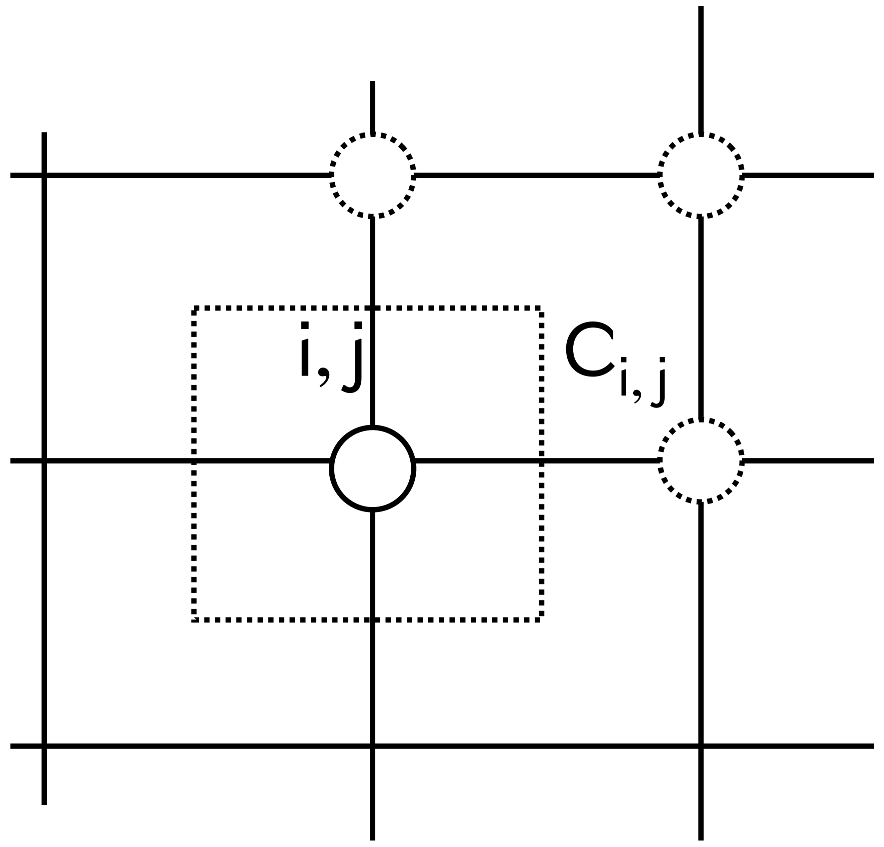
Residual distribution and global fluxes : Cartesian grids

STEP 1: write equivalent consistent fluxes for RD in homogeneous case

(R. Abgrall and MR, ECM 2017, Abgrall CMAM 2018)

$$|C_{i,j}|\frac{du_{i,j}}{dt} + \sum_{C \ni i,j} \phi_{i,j} = 0$$





Equate all $\phi_{i,j}$ and $\sum (\hat{F}_{n_{v_{ij}}} - F_{ij} \cdot \hat{n}_{v_{ij}})$

Add constraints

Solve linear system for $\hat{F}_{n_{v_{ij}}}$

$$|C_{i,j}|\frac{du_{i,j}}{dt} + \sum_{C \ni i,j} \sum_{v_{ij}} (\hat{F}_{n_{v_{ij}}} - F_{ij} \cdot \hat{n}_{v_{ij}}) = 0$$

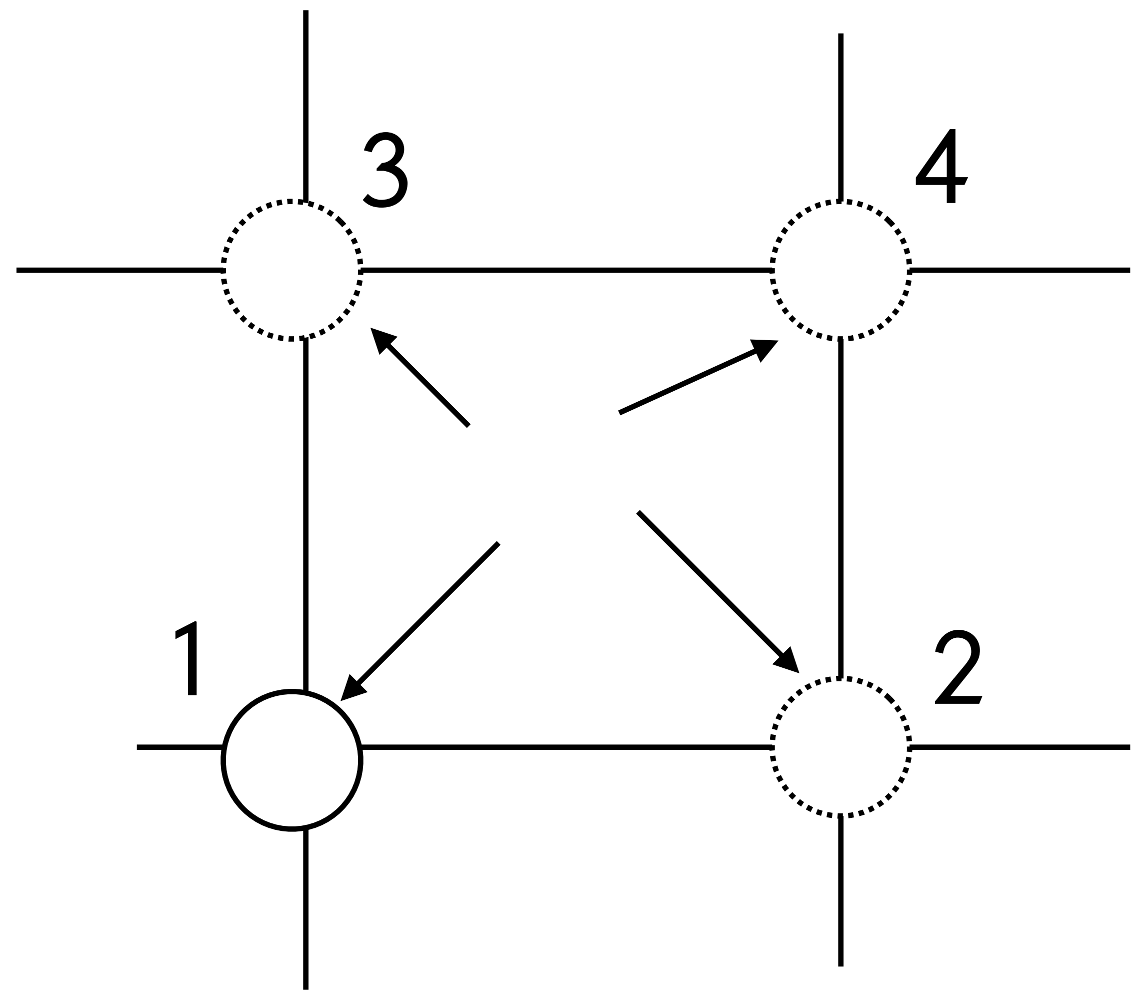
Closing(1). Going multiD

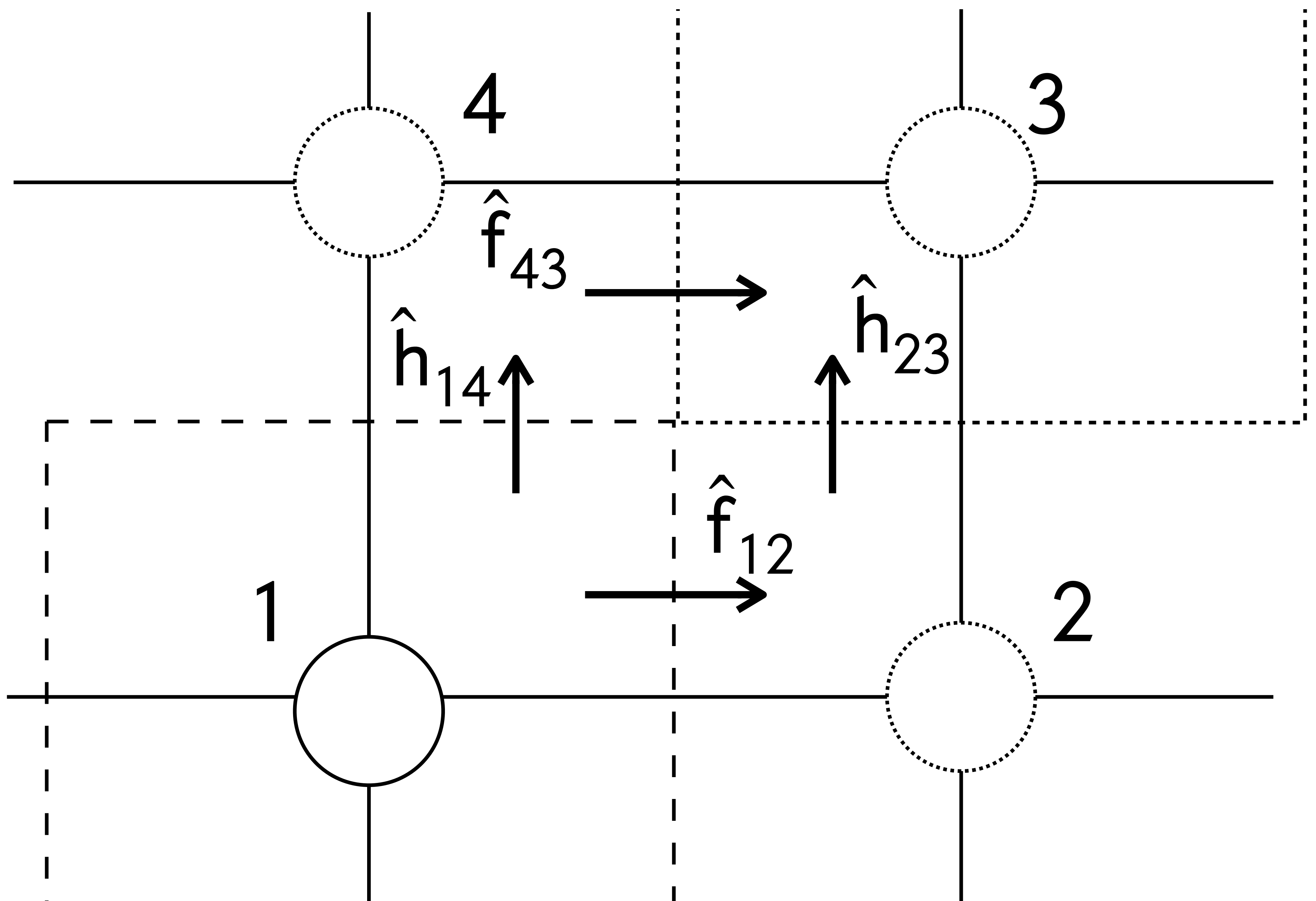
Consistent fluxes : Cartesian grids

$$\hat{F}_{n_{ij}} = \sum_{l=1,4} \alpha_l \psi_l \quad \text{with} \quad \sum_{l=1,4} \alpha_l = 1$$

$$\psi_l = F_l \cdot \overrightarrow{n}_l + \phi_l$$

Diagonal leaving l





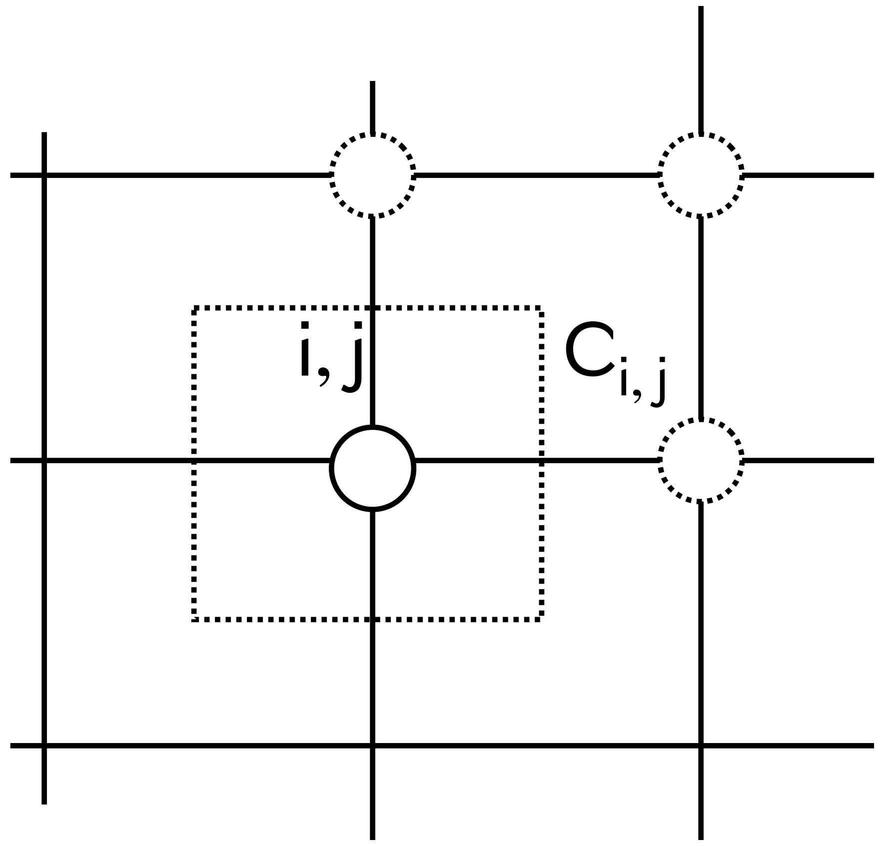
Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) = 0$$

Residual distribution and global fluxes : Cartesian grids

STEP 1: write equivalent consistent fluxes for RD in homogeneous case

(R. Abgrall and MR, ECM 2017, Abgrall CMAM 2018)



$$|C_{i,j}| \frac{du_{i,j}}{dt} = -\sum_{C \ni i,j} \phi_{i,j} = -\sum_{C \ni i,j} \sum_{v_{ij}} \hat{F}_{n_{v_{ij}}}$$

$\hat{F}_{n_{v_{ij}}} = F \cdot n_{v_{ij}}$ for **u** <u>constant over the cell (multiD)</u>

Notion of consistency wrt constants

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

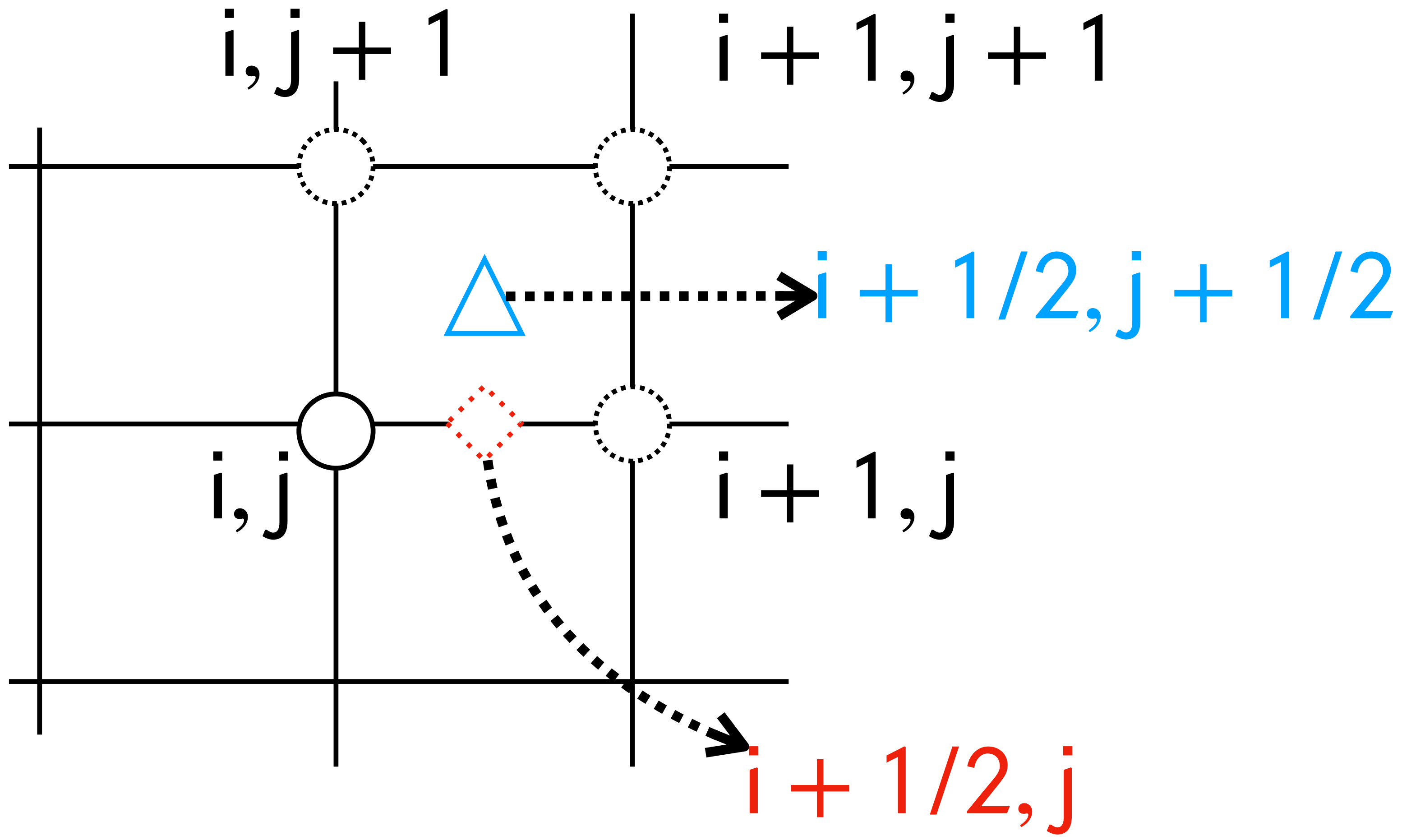
Residual distribution and global fluxes : Cartesian grids

STEP 2: what is the source integral equal to in 2D ?

$$|C_{i,j}| \frac{du_{i,j}}{dt} + \sum_{C \ni i,j} \phi^C_{i,j} = 0$$

$$\sum_{i,j \in C} \phi^C_{i,j} = \phi^C = \int_C (\nabla \cdot F + S)$$

Closing(1). Going multiD



$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

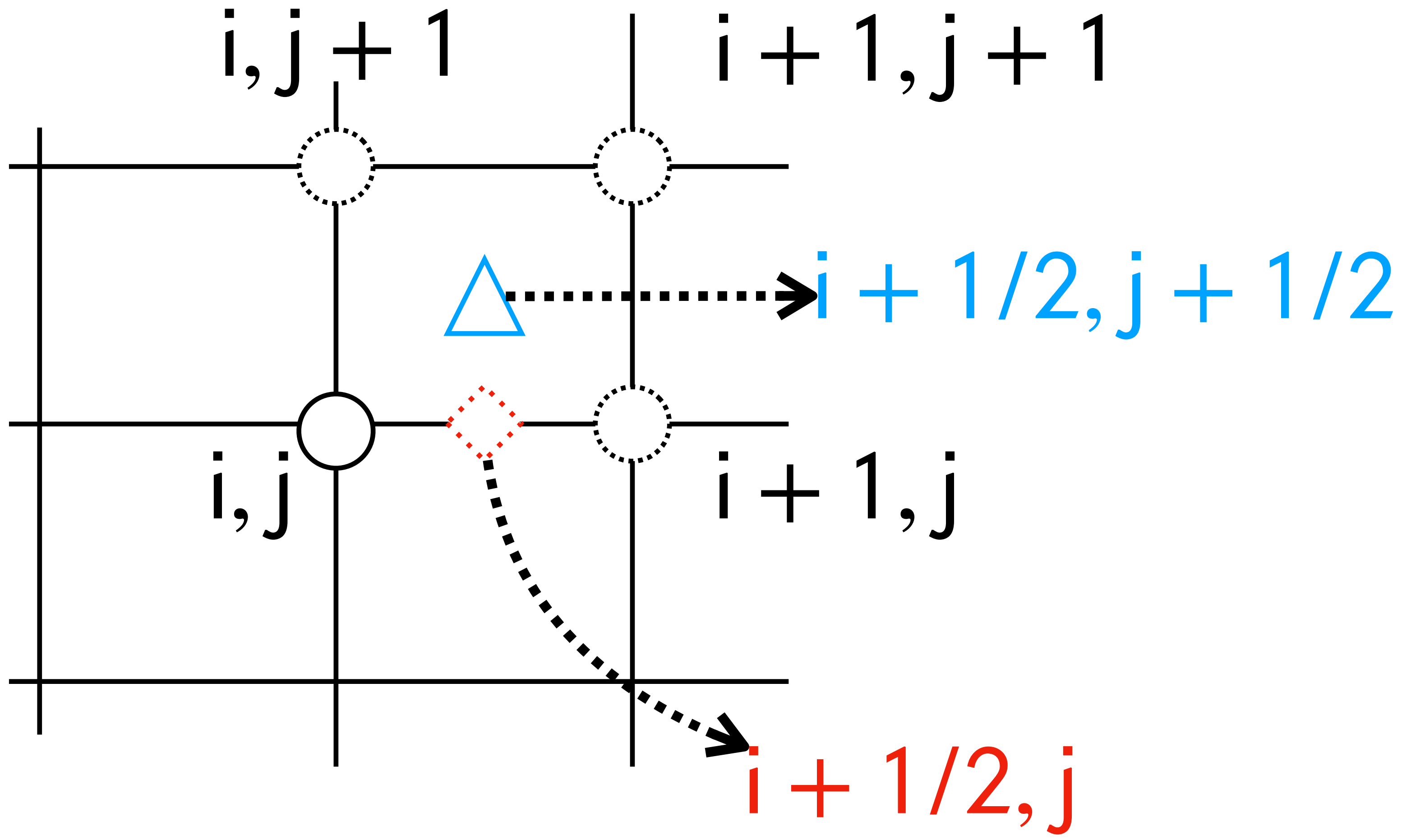
Residual distribution and global fluxes : Cartesian grids

STEP 2: what is the source integral equal to in 2D ?

$$\int_{C_{i+1/2,j+1/2}} \nabla \cdot F = \Delta y(f_{i+1,j+1/2} - f_{i,j+1/2}) + \Delta x(h_{i+1/2,j+1} - h_{i+1/2,j})$$

$$\int_{C_{i+1/2,j+1/2}} \nabla \cdot \sigma \stackrel{?}{=} \int_{C_{i+1/2,j+1/2}} S$$

Closing(1). Going multiD



$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Residual distribution and global fluxes : Cartesian grids

STEP 2: what is the source integral equal to in 2D ?

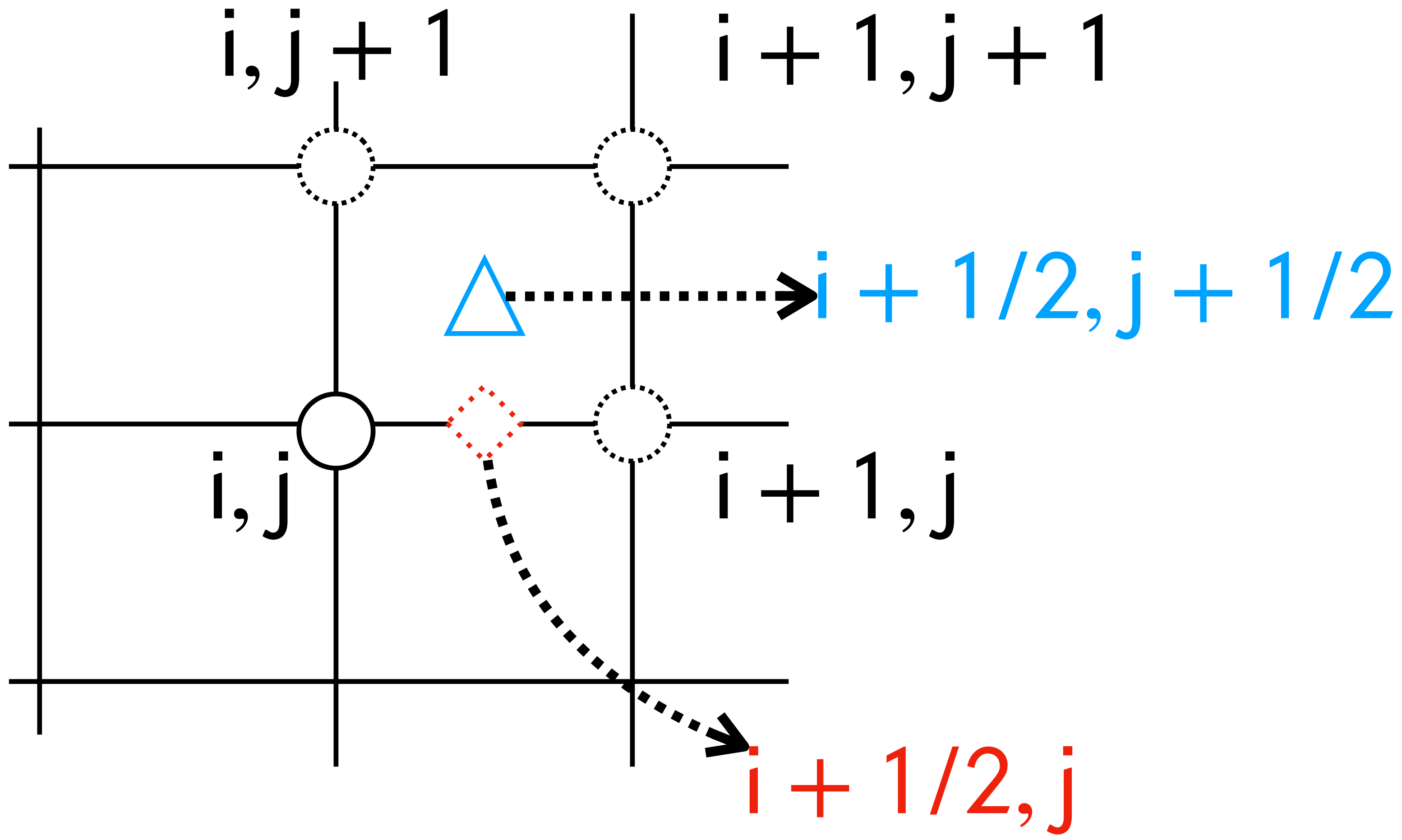
Define for example the edge fluxes

$$\sigma^x_{i+1,j+1/2} = \sigma^x_{i,j+1/2} + \frac{|C_{i+1/2,j+1/2}|}{2\Delta y} S_{i+1/2,j+1/2} \qquad \sigma^y_{i+1/2,j+1} = \sigma^y_{i+1/2,j} + \frac{|C_{i+1/2,j+1/2}|}{2\Delta x} S_{i+1/2,j+1/2}$$

To get

$$\int_{C_{i+1/2,j+1/2}} \nabla \cdot \sigma = \int_{C_{i+1/2,j+1/2}} S$$

Closing(1). Going multiD



$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$

Residual distribution and global fluxes : Cartesian grids

STEP 2: what is the source integral equal to in 2D ?

As before we can set

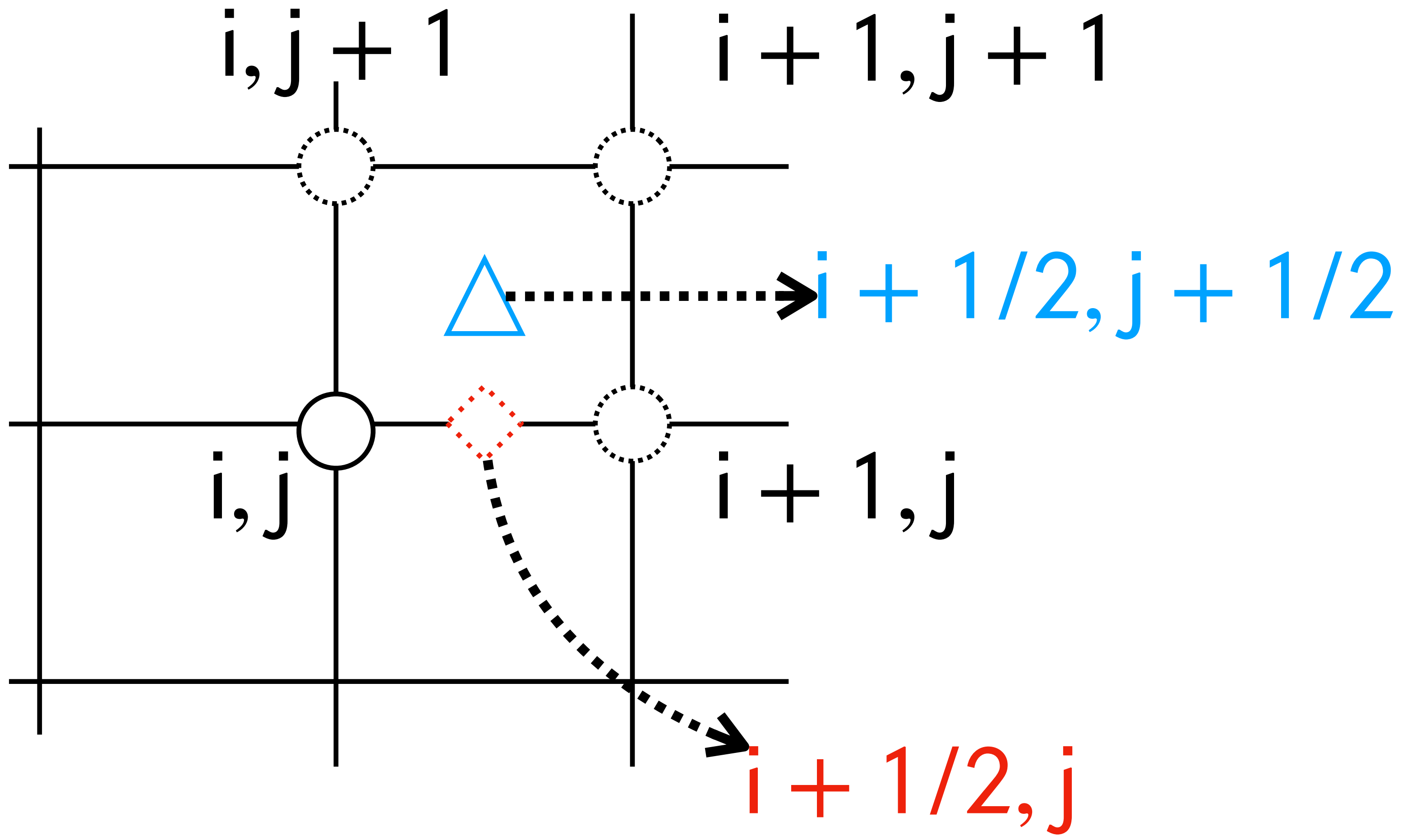
$$-\Delta x\, \sigma^x_{i,j+1/2} = \psi_{i+1/2,j+1/2} - \psi_{i-1/2,j+1/2} \qquad -\Delta y\, \sigma^y_{i+1/2,j} = \psi_{i+1/2,j+1/2} - \psi_{i+1/2,j-1/2}$$

To show

$$-\frac{\psi_{i+3/2,j+1/2} - 2\psi_{i+1/2,j+1/2} + \psi_{i-1/2,j+1/2}}{\Delta x^2} - \frac{\psi_{i+1/2,j+3/2} - 2\psi_{i+1/2,j+1/2} + \psi_{i+1/2,j-1/2}}{\Delta y^2} = S_{i+1/2,j+1/2}$$

Closing(1). Going multiD

$$\partial_t u + \nabla \cdot F(u) + S(u; d) = 0$$



Residual distribution and global fluxes : Cartesian grids

STEP 2: what is the source integral equal to in 2D ?

There are many different possibilities

- Edge fluxes (using cell source values) - cell potentials
- Cell flux values (using edge averaged source values) - edge potentials
- unstaggered with nodal fluxes and potentials
- etc

These can be all embedded in the residual distribution formalism

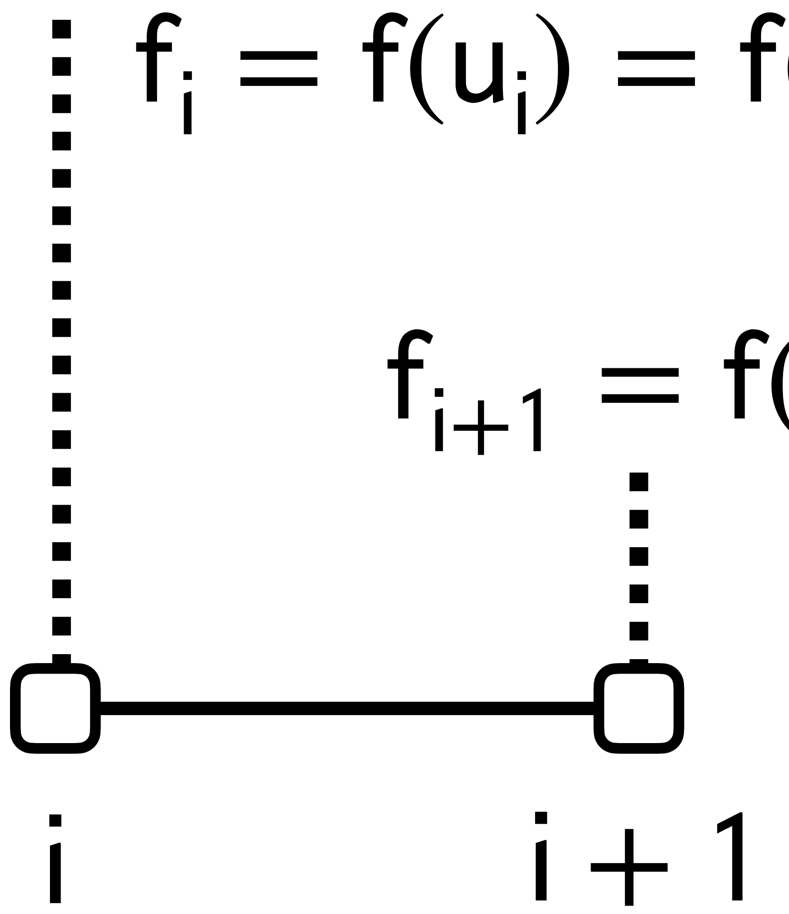
Closing(2). Final boss: subgrid structure

Residual Distribution

$$\phi^{i+1/2} = \int_{\Delta_{i+1/2}x} (\partial_x f + S) = \Delta f + \int_{\Delta_{i+1/2}x} S$$

$$f_i = f(u_i) = f(v(u_i; d_i))$$

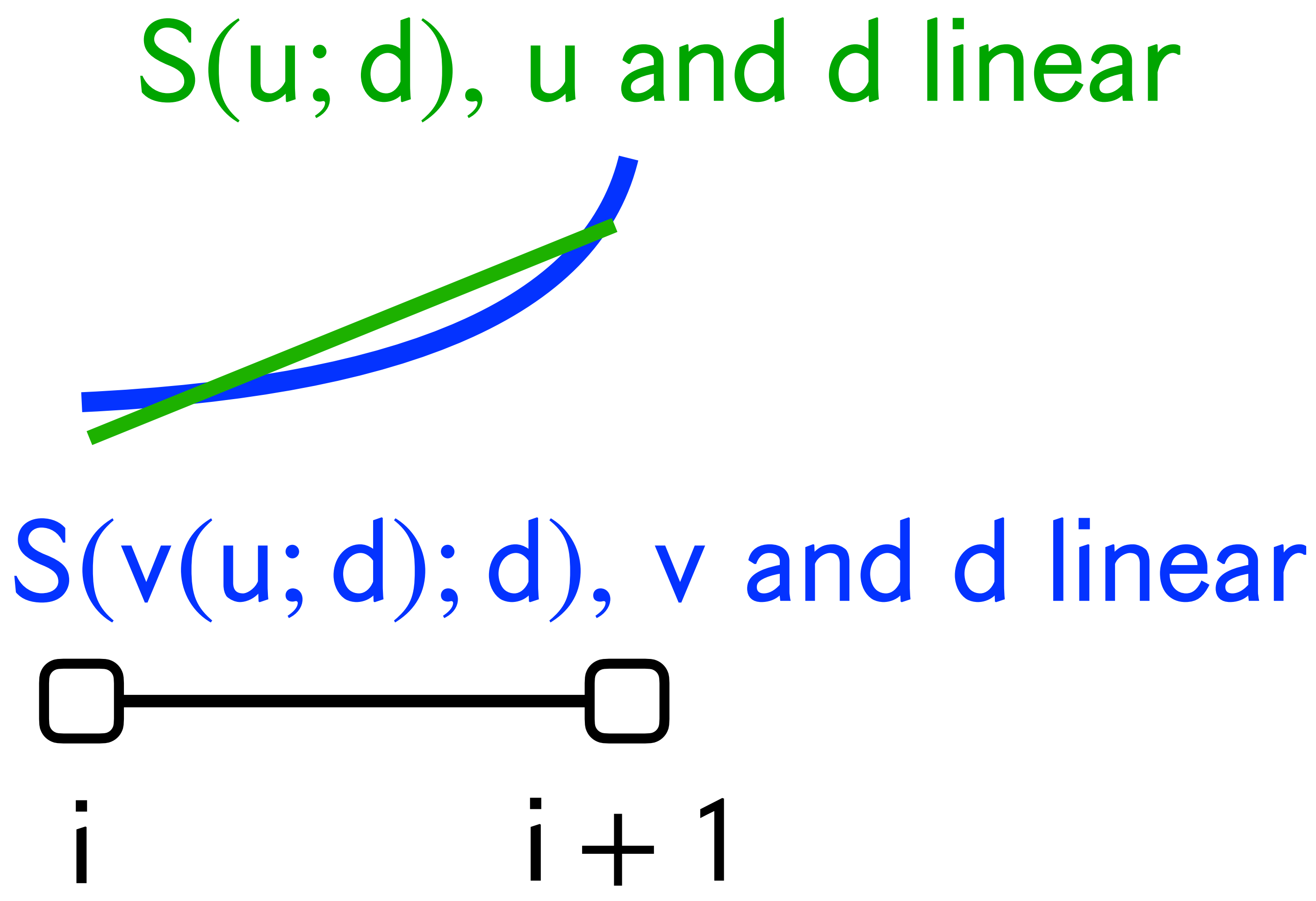
$$f_{i+1} = f(u_{i+1}) = f(v(u_{i+1}; d_{i+1}))$$



$$\forall v = v(u; d)$$

Global Flux

$$g_i := f_i + \int_{\Delta_{i+1/2}x} S$$



there are $N \gg 1$ possibilities

Closing(2). Final boss: subgrid structure

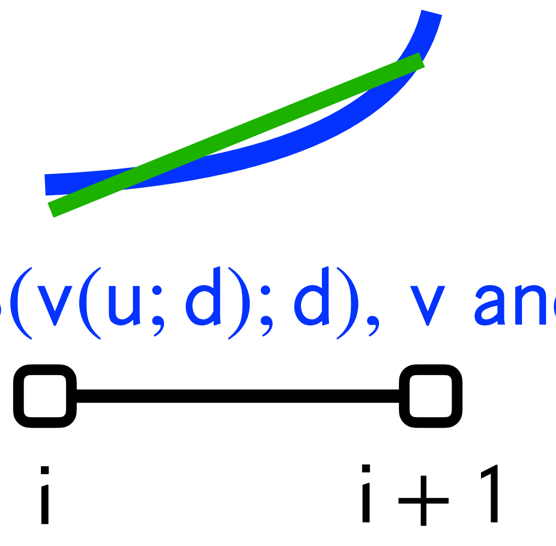
Residual Distribution

$$\phi^{i+1/2} = \int_{\Delta_{i+1/2}x} (\partial_x f + S) = \Delta f + \int_{\Delta_{i+1/2}x} S$$

$$\phi_i^{i+1/2} = \frac{\phi^{i+1/2}}{2} + \text{Stab}_i$$

$$\text{Stab}_i = \begin{cases} \int_{\Delta_{i+1/2}x} \gamma(\partial_x f + S) \\ \alpha_{i+1/2}\Delta_{i+1/2}u \\ \theta_i [\![u_x]\!]_i + \theta_{i+1}[\![u_x]\!]_{i+1} \\ \text{etc} \end{cases}$$

S(u; d), u and d linear



there are $N \gg 1$ possibilities

Global Flux

$$g_i := f_i + \int_{\Delta_{i+1/2}x} S$$

$$g_{i+1/2} = \frac{g_{i+1} + g_i}{2} + \text{Stab}_{i+1/2}$$

$$\text{Stab}_{i+1/2} = \alpha_{i+1/2} \begin{cases} \Delta_{i+1/2}u \\ (v_u)^{-1}_{i+1/2}\, \Delta_{i+1/2}v \\ (g_u)^{-1}_{i+1/2}\, \Delta_{i+1/2}g \\ \text{etc} \end{cases}$$

Closing(2). Final boss: subgrid structure

Perturbation of the constant slope eq.: $gh\nabla b = -c_f(h, \|\vec{u}\|)\vec{u}$

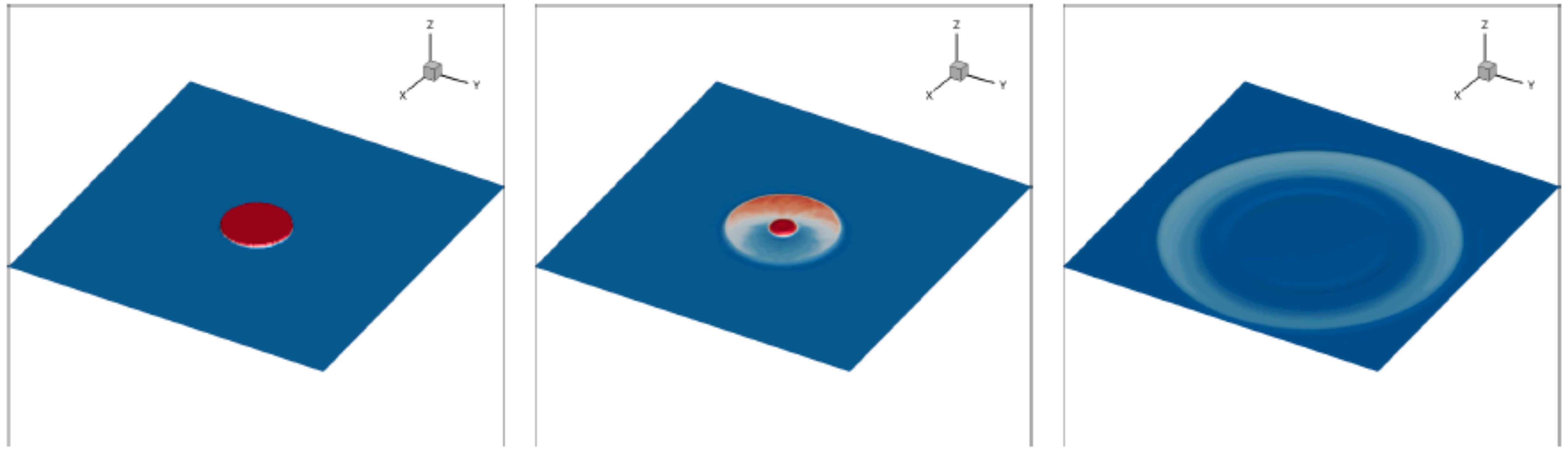


Figure 5.10: Sub-critical flow in slopping channel with friction at different times for a Manning coefficient of 0, 5 with the IMEX RK RD scheme.

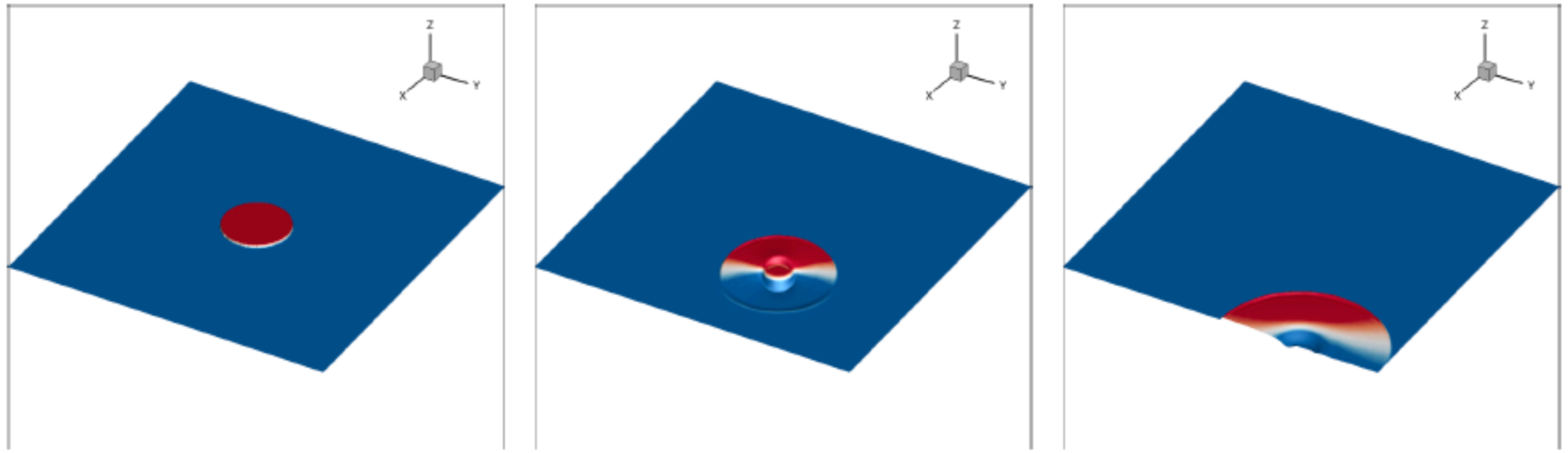


Figure 5.12: Super-critical flow in slopping channel with friction at different times for a Manning coefficient of 0, 02 with the IMEX RK RD scheme.

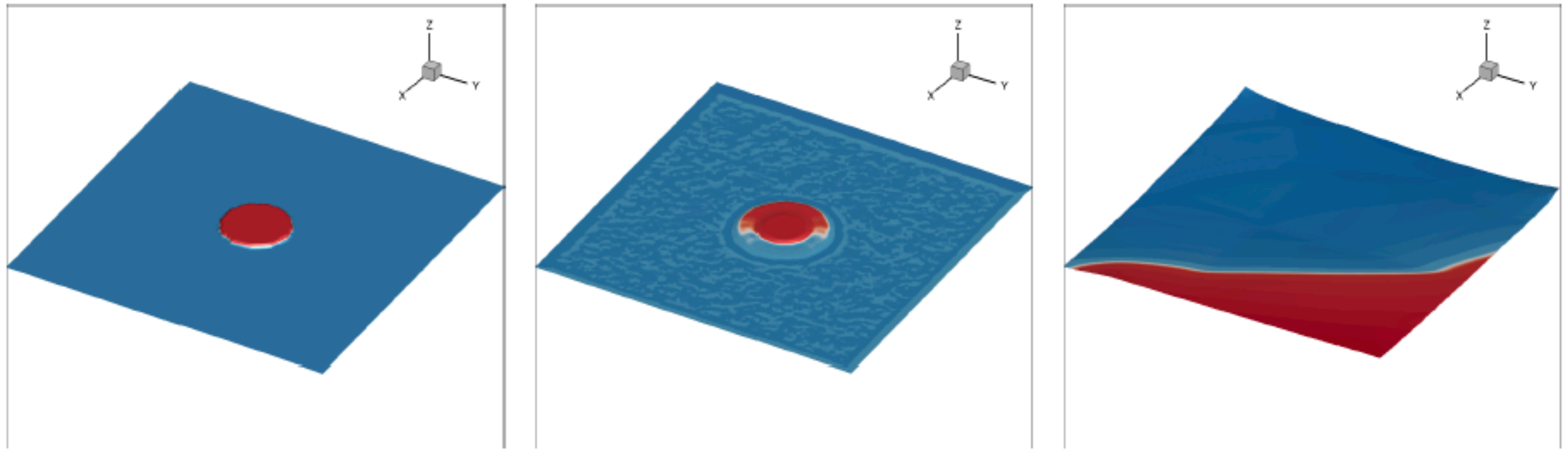
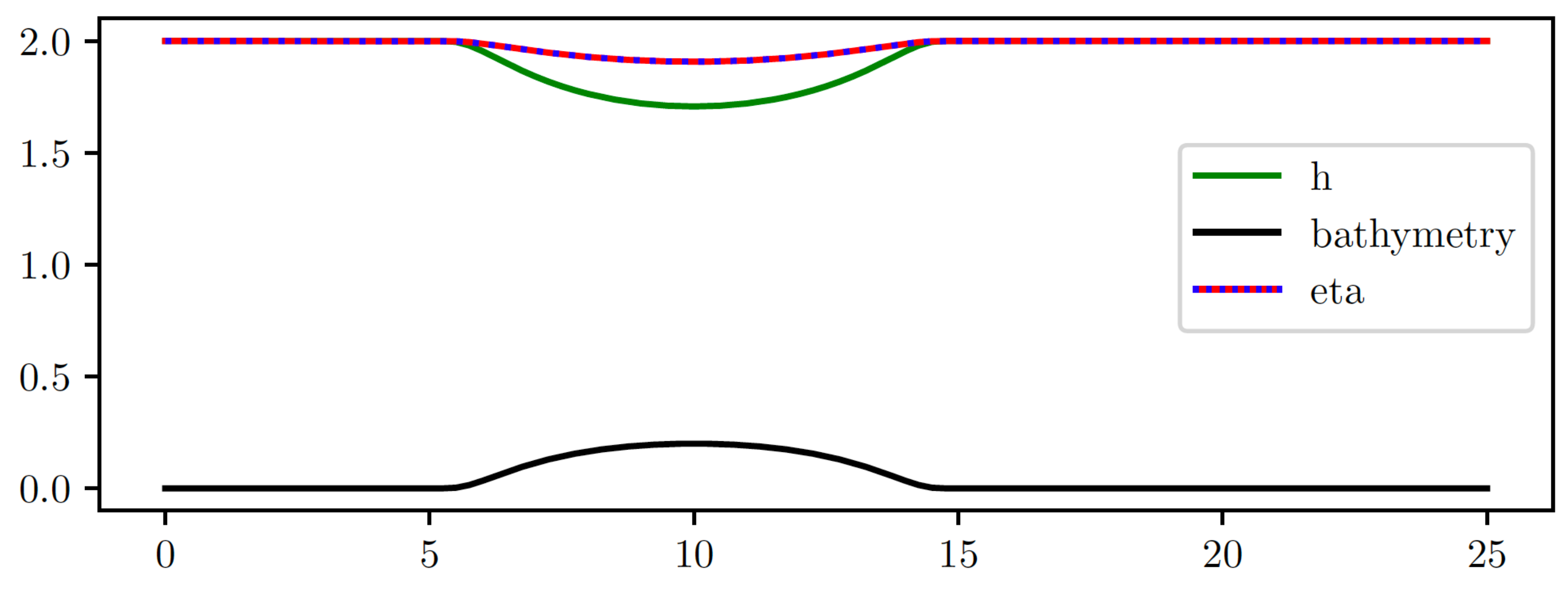
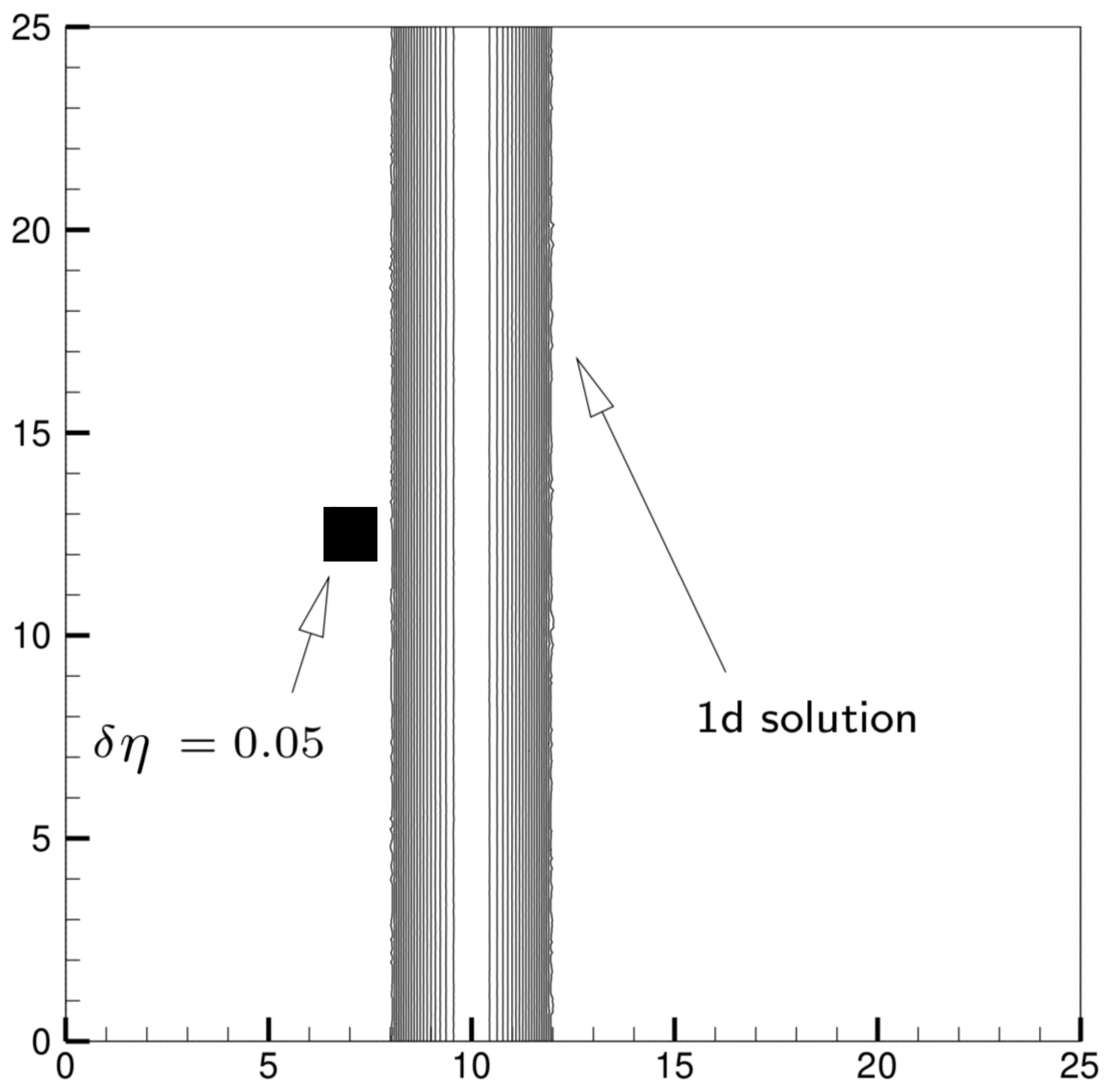


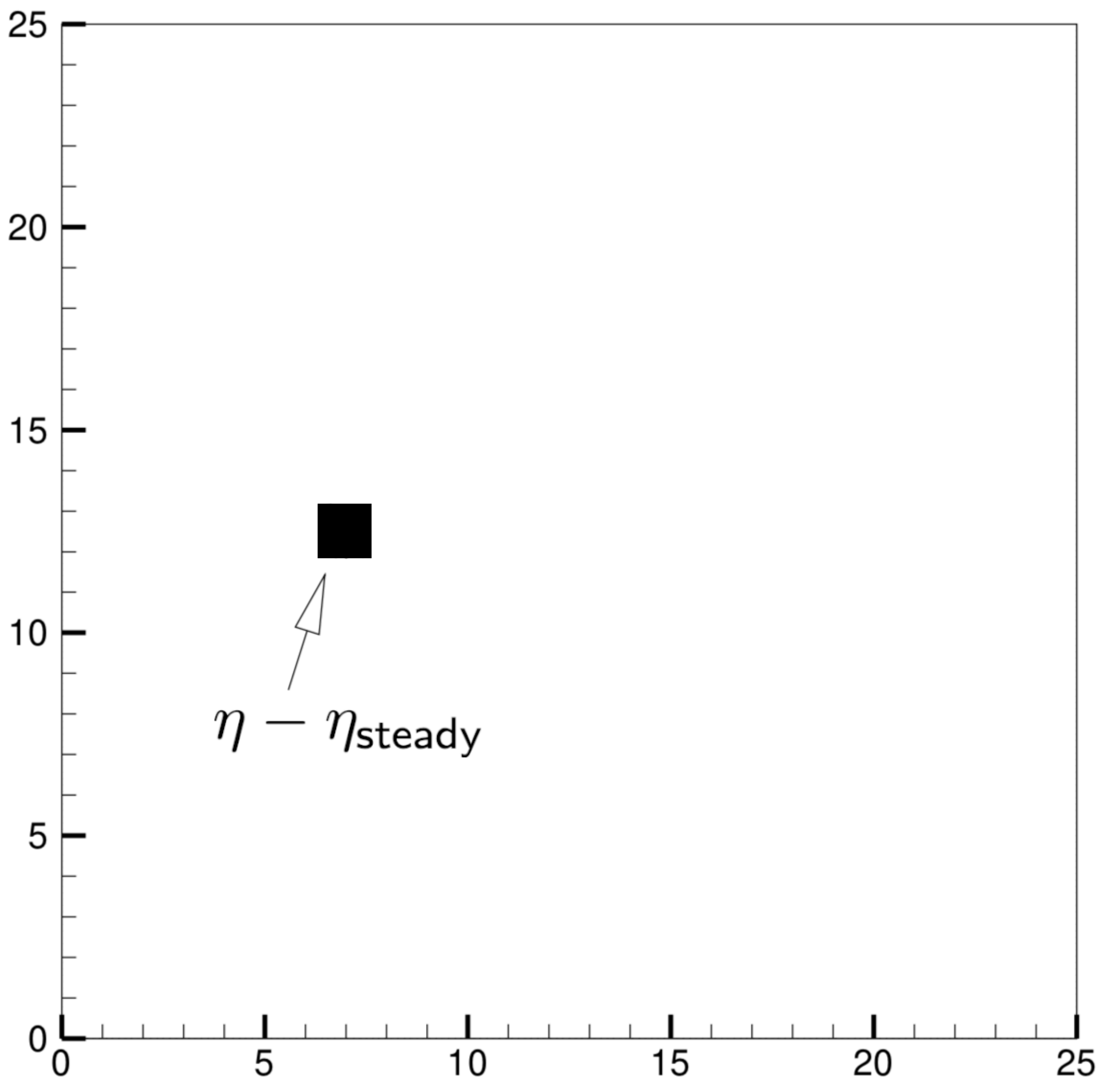
Figure 5.13: Flow in slopping channel with friction with a perturbation at different times for a Manning coefficient of 0, 2 with the Strang splitting RD scheme.

Closing(2). Final boss: subgrid structure

Perturbation of the classical subcritical case



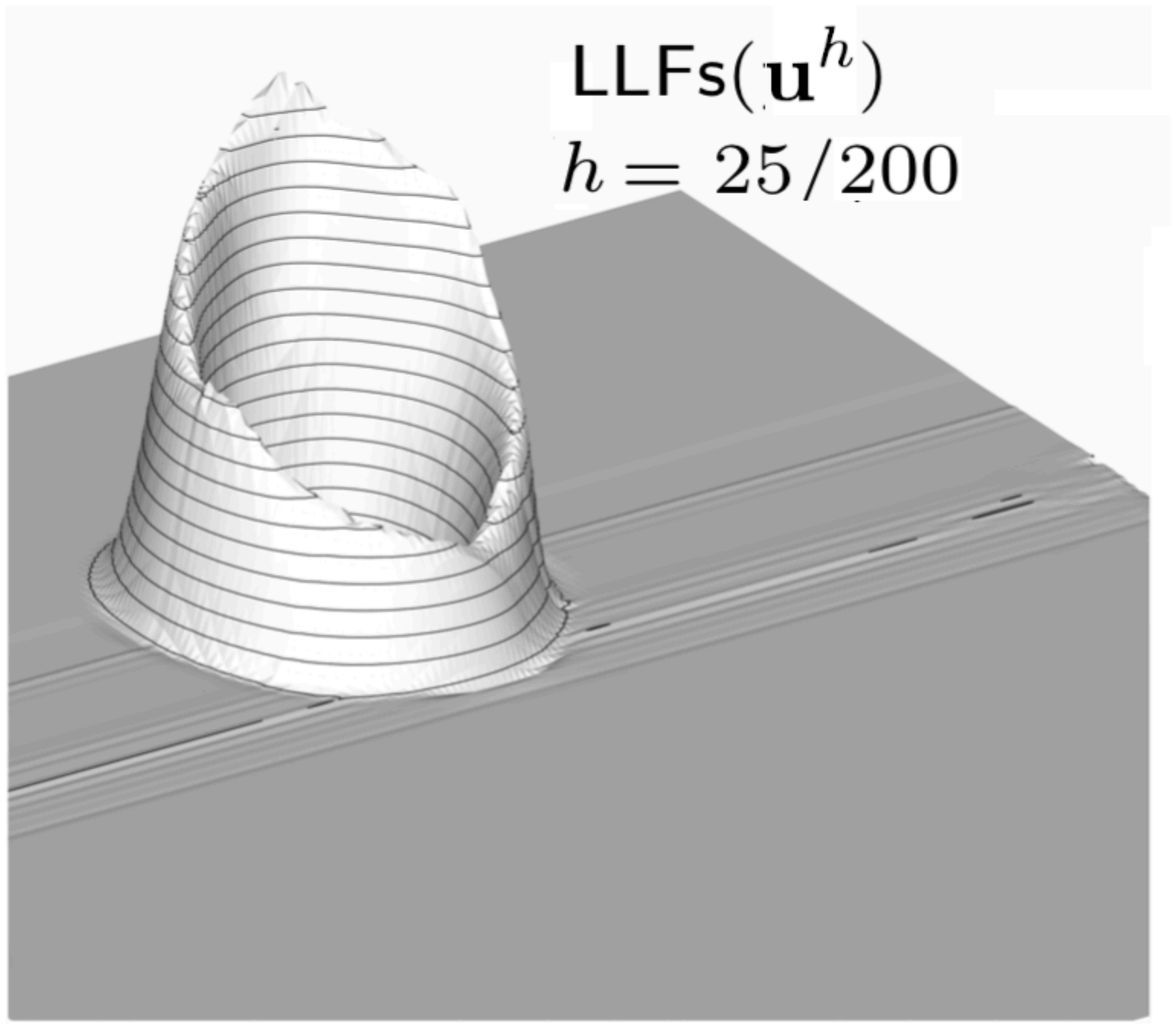


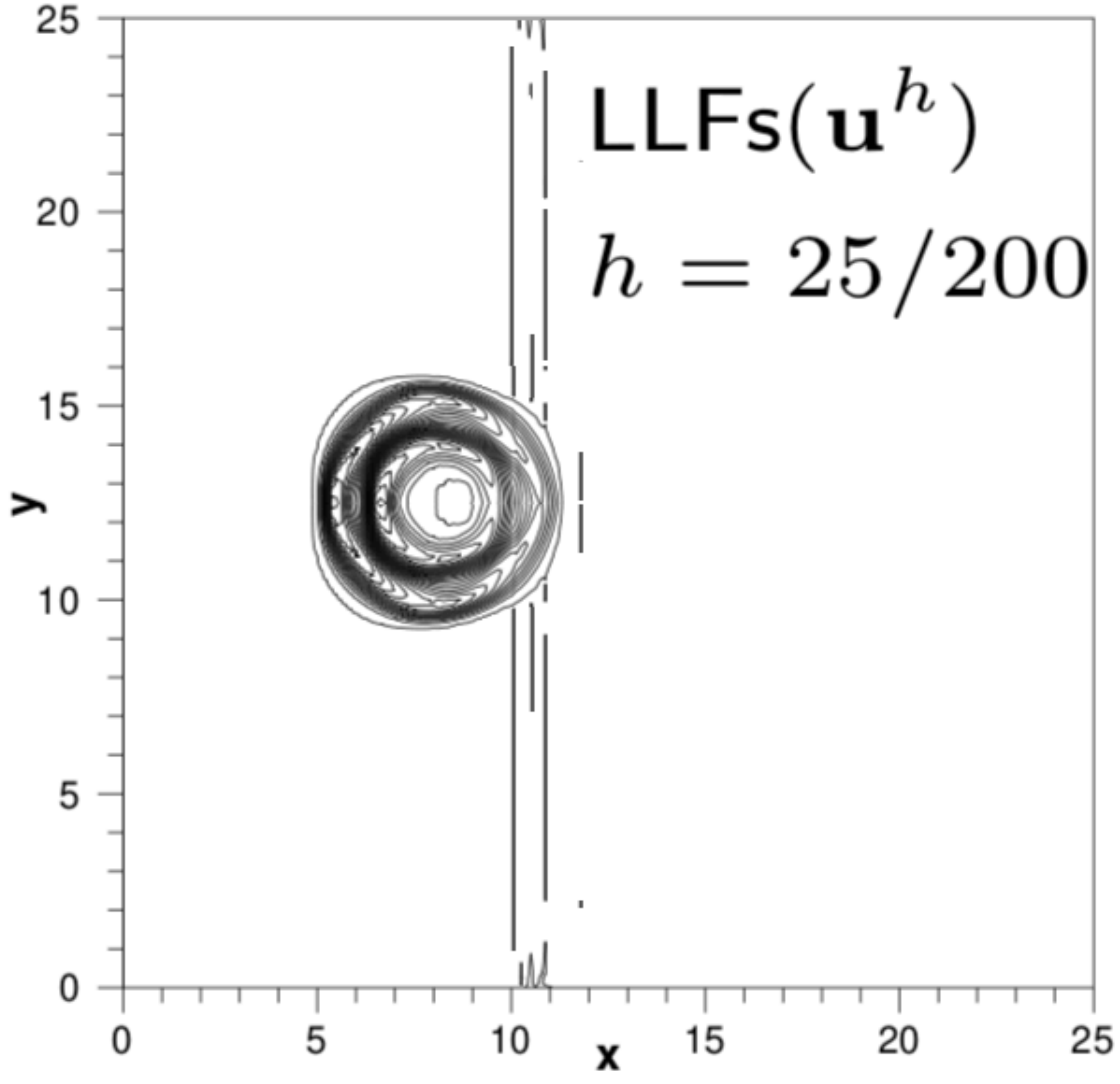


Delestre et al, IJNMF 2013

Closing(2). Final boss: subgrid structure

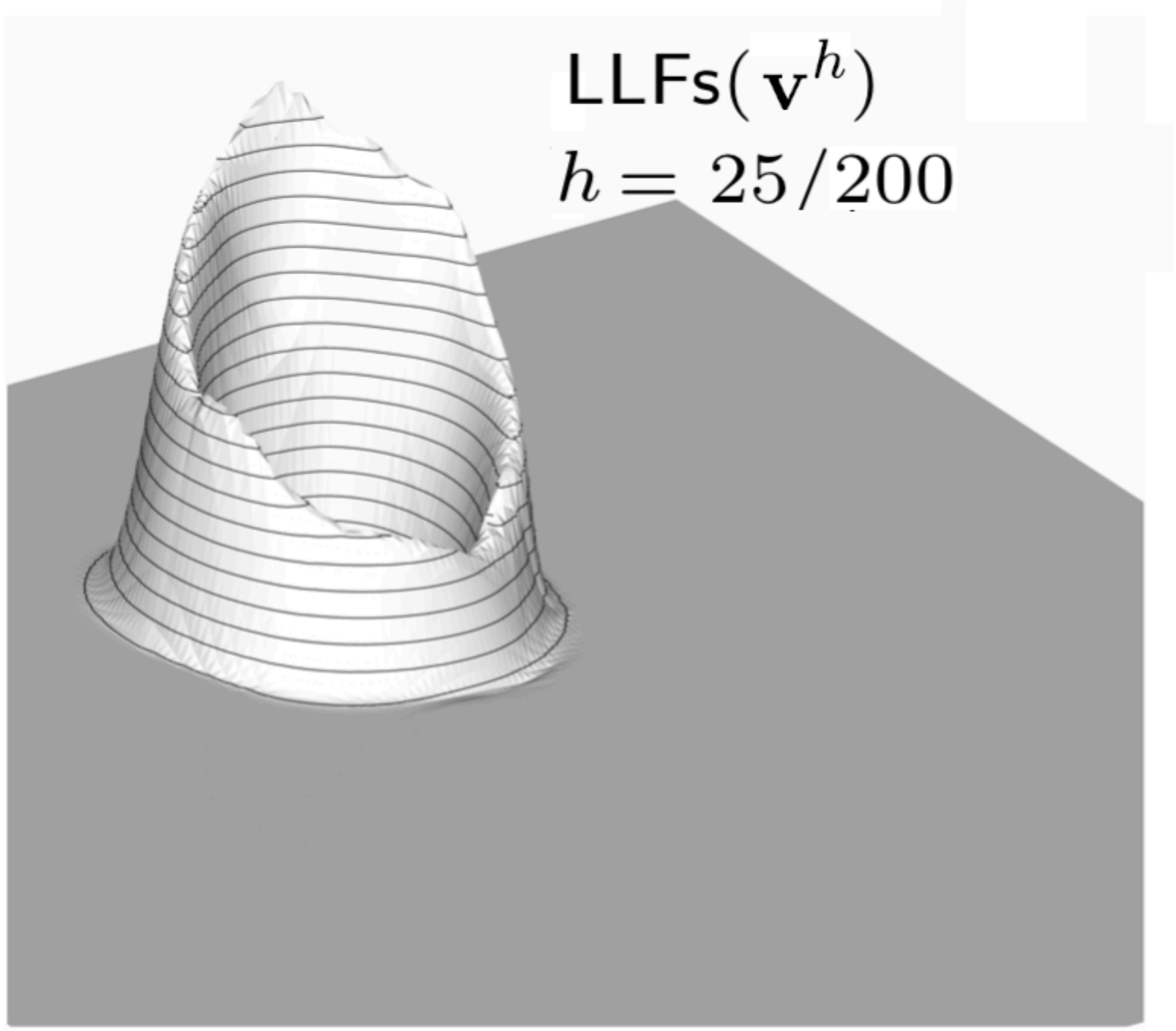
Perturbation of the classical subcritical case

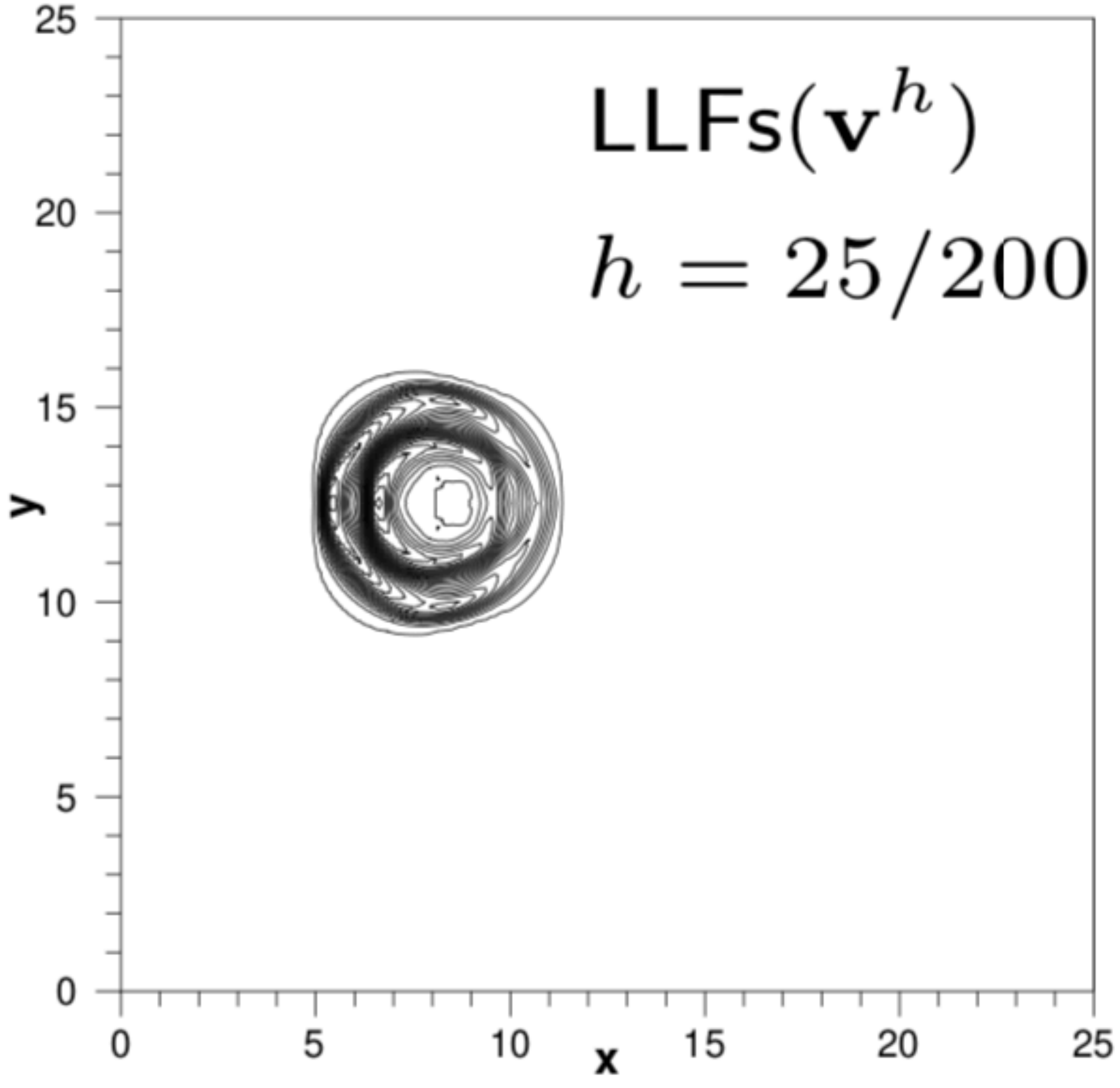




Closing(2). Final boss: subgrid structure

Perturbation of the classical subcritical case





Final summary

Error balance and residual distribution/fluctuation splitting

- balance laws —> equivalence with FV and **consistent global fluxes**
- unsteady: global flux and mass matrix, time stepping issues
- variety of source terms and so far no a-priori knowledge on the ex. sol !
- going multiD : global flux and solenoidal involutions —> more on potentials ?
- sub-grid structure:
  - invariants
  - residual based/global flux for improved consistency (not always exactly WB)
  - residual/global flux based stabilization to preserve consistency
  - initial work by Castro and Pares to “guess" the sub-grid structure (inverse pb)
  - more to come for sure …

Inria

THX