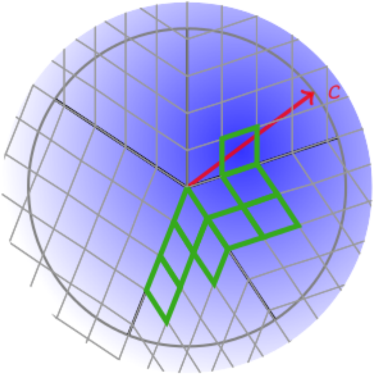
Geometric Random Edge

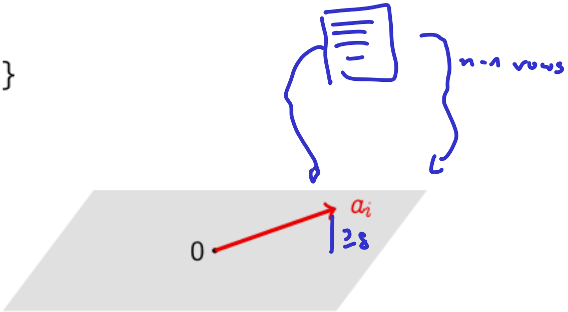


Friedrich Eisenbrand
November 21, 2014

The δ-distance property

$$\max\{c^T x \colon Ax \leqslant b\}$$

$\|\cdot\| = 1$



- Suppose each row a_i of A satisfies $||a_i|| = 1$
- Distance of row to subspace generated by other rows is $\geq \delta$
- *δ-distance property*

(Brunsch & Röglin 2013)

The δ-distance property

$D \leq p(n, 1/\delta)$

$\max\{c^T x \colon Ax \leqslant b\}$

- Suppose each row a_i of A satisfies $||a_i|| = 1$
- Distance of row to subspace generated by other rows is $\geq \delta$
- *δ-distance property*

(Brunsch & Röglin 2013)

Knowing element of optimal basis

$\max\{c^T x : Ax \le b\} = \max c^T U U^{-1} y : A U U^{-1} y \le b$

$B \subseteq \{1, \dots, m\}$

$U^{-1} x = y$

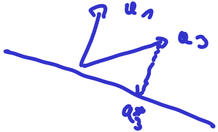
- a_1 element of optimal basis
- Rotate into e_1

$$a_1^T U = e_1^T$$

- $\max\{c^T Ux : AUx \le b\}$

$x_1 \le b_1$ tight.

$x_1 < b_1$



$\begin{pmatrix} a_2^* \\ \vdots \\ a_m^* \end{pmatrix}$ δ-distance property.

Lemma

If $a_1, \dots, a_m$ satisfy δ-distance prop. then $\boxed{a_2^*, \dots, a_m^*}$ satisfy δ-distance prop.

$d(a_2^*, \langle a_3^*, \dots, a_n^* \rangle)$

$\ge d(a_2^*, \langle a_1, a_3^*, \dots, a_n^* \rangle)$

$= d(a_i^*, \langle a_1, a_2, \dots, a_n \rangle) = \ge \delta$

$d(a_i, \langle a_1, \dots, a_n \rangle)$

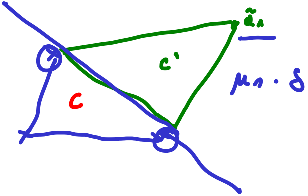
Close to c

$c' = \underline{\mu_1 \tilde{a}_1} + \mu_2 \tilde{a}_2 + \mu_3 \tilde{a}_3$

Lemma

$B \subseteq \{1, \ldots, m\}$ optimal basis of the LP, B' be an optimal basis of LP with c being replaced by c'. If

$$c' = \sum_{j \in B'} \mu_j a_j .$$



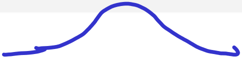
then for $k \in B' \setminus B$, one has

$\|c - c'\| \leq \frac{\delta}{n}$

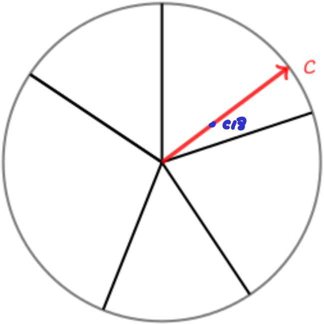
$$\|c - c'\| \geqslant \delta \cdot \mu_k .$$

Element of opt basis of c' with largest weight $\Rightarrow$ in the optimal basis of LP

Sampling a point w.r.t Gaussian



- Consider Gaussian
  $g(x) = \exp(-||x - c/8||^2/\boxed{(2t_0)})$



Sampling a point w.r.t Gaussian

- Consider Gaussian
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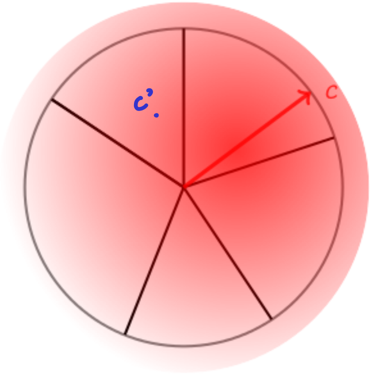
- Sample c' according to Gausssian



Sampling a point w.r.t Gaussian

- Consider Gaussian $g(x) = \exp(-||x - c/8||^2/(2t_0))$
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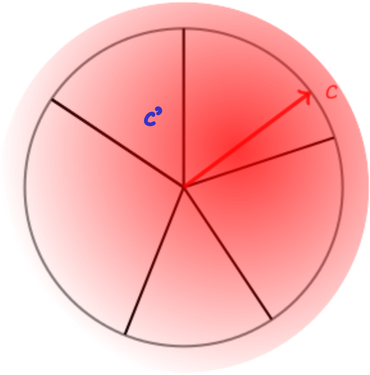
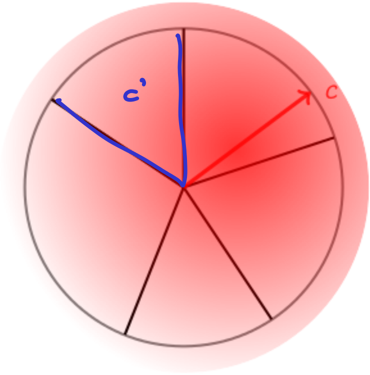
- If $t_0 \approx \delta^2/n^3$ then $\boxed{||c - c'|| \leq \delta/(2n)}$ W.H.P.



Sampling a point w.r.t Gaussian

- Consider Gaussian $g(x) = \exp(-||x - c/8||^2/(2t_0))$
- Sample c' according to Gausssian
- If $t \approx \delta^2/n^3$ then $||c - c'|| \leq \delta/(2n)$ W.H.P.
- Idea: Random walk keeps track of pivots with stationary distribution $\approx g(x)$

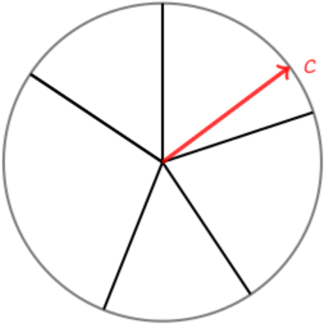


Walking in the space of cones

If we

- start within cone of feasible solution
- leave a cone only through facet
- do not cross cones in one step

then we can keep track of optimal basis.



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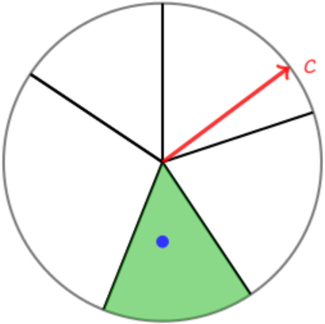

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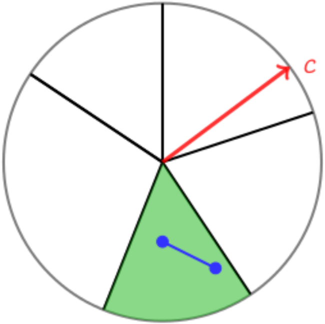

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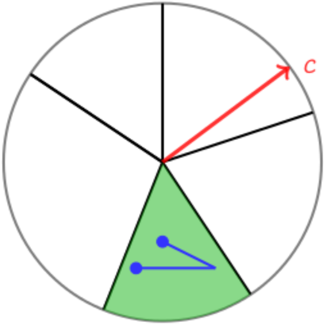

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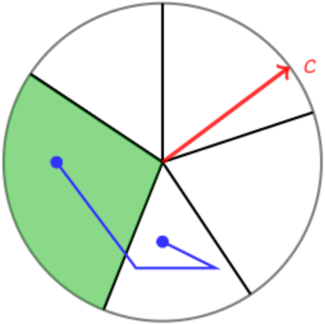

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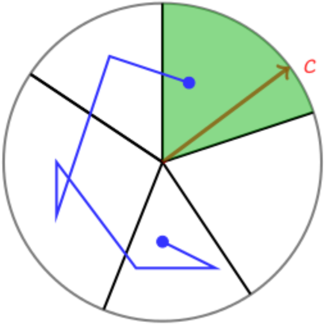
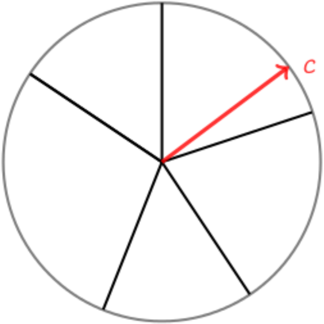


The random walk

- Partition space of cones into small parallelepipeds, as in (Dyer and Frieze 1994)



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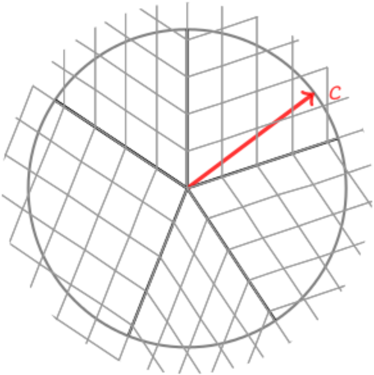

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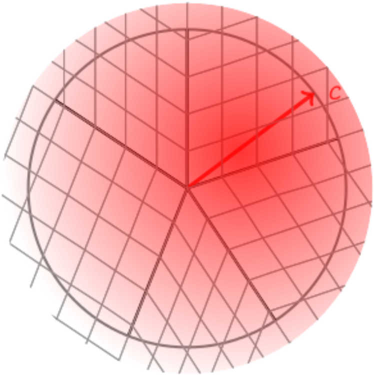

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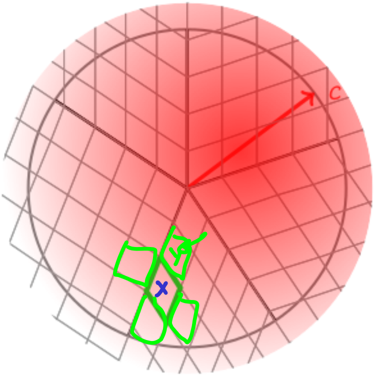
- At given parallelepiped X:
  - With probability 1/2 don't do anything (lazy!)
  - Choose neighbor Y uniformly at random
  - Make transition with prob. $\min\{1, \mu(Y)/\mu(X)\}$

$$\frac{\mu(x)}{\sum_{y} \mu(y)}$$



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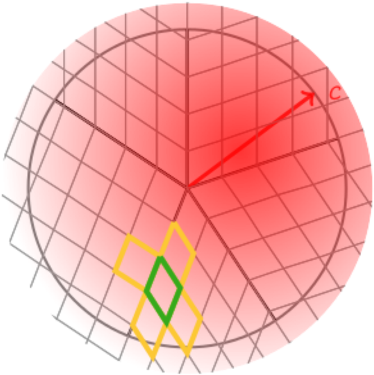

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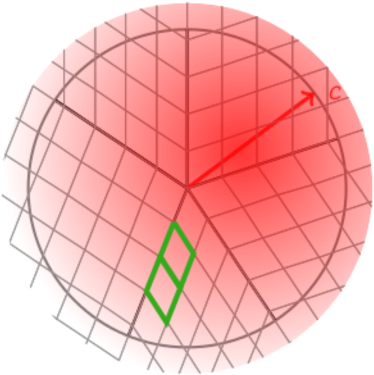

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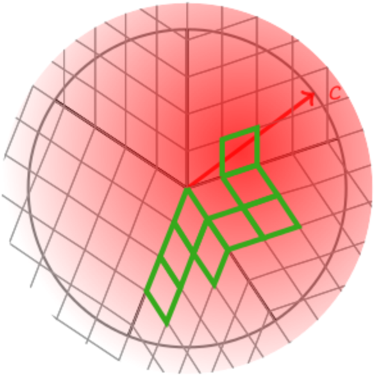
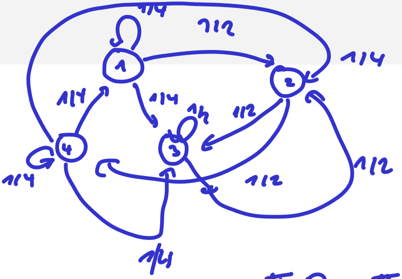


Some basics in Markov chains

$|\text{Eigenvalues}| \leq 1$



- $P \in \mathbb{R}_{\geq 0}^{V \times V}$ transition matrix
- Distribution: $\pi \in \mathbb{R}_{\geq 0}^{V}$ with $\pi \mathbf{1} = 1$
- After t steps: $\boxed{\pi^{(t)}} = \underline{\pi P^t}$
- Stationary distribution: $\boxed{\pi_s P = \pi_s}$ Eigenvector w. EV 1.
- Questions: Does π_s exist? Is it unique? Does random walk *converge to π_s and how fast?*

$\pi P = \pi'$

$P = \begin{array}{c|cccc} 1 & 1/4 & 1/2 & 1/4 & 0 \\ 2 & & & & \end{array} \leftarrow \Sigma = 1$

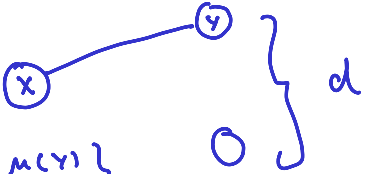
$\pi P = 2 \cdot \pi$

$\pi \cdot P^t = \underline{2^t \cdot \pi}$

The lazy Markov chain with Metropolis filter

$$\boxed{Q(x)\cdot p(x,y) = Q(y)\cdot p(y,x)}$$

- Time reversible!
- Stationary distribution: $\mu(X)/\sum_Y \mu(Y)$



$$\mu(x)\cdot p(x,y) = \mu(y)\cdot p(y,x)$$

$$\min\left\{1, \frac{\mu(Y)}{\mu(X)}\right\}$$

$$\frac{\mu(x)}{\sum \mu(y)} \overset{?}{=} \sum_{y\in N(x)} \frac{Q(y)}{\Sigma}\cdot p(y,x) + \frac{\mu(x)}{\Sigma}\left(1-\sum_{y\in N(x)} p(x,y)\right) = \frac{\mu(x)}{\Sigma}$$

$$\mu(x)\cdot \min\left\{1, \frac{\mu(y)}{\mu(x)}\right\}\cdot \frac{1}{2d} = \frac{\min\{\mu(x), \mu(y)\}}{2d} = \mu(y)\cdot p(y,x)$$

Markov chains: Stationary distribution and Eigenvalues

connected

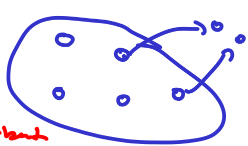
$\pi(P-I)=0 \qquad (P-I)\mathbb{1}=0$

- π_s is left Eigenvector with Eigenvalue 1
- All Eigenvalues are $|\cdot| \leq 1$
- Uniqueness: Dimension of Eigenspace for 1 is one.
- rank$(P-I)$ is $V-1$: Otherwise $\underline{\exists v \perp 1}$ with $\underline{Pv = v}$

rank$(P-I) = n-1$

Suppose rank$(P-I) \leq n-2$

$v \neq 0 \qquad P \cdot v = v, \quad v \perp \mathbb{1}$



uniqueness of stationary distribution

The lazy case: $\lambda_{min} \geqslant 0$

$$\begin{pmatrix} \geq 1/2 & & \\ & \ddots & \\ & & \geq 1/2 \end{pmatrix}$$

- Suppose all diagonal entries of P are $\geq \gamma$
- $P = (1-\gamma)\tilde{P} + (1-(1-\gamma))I$ with

$$\boxed{\tilde{P} = \frac{1}{1-\gamma}P + \left(1 - \frac{1}{1-\gamma}\right) I}$$

$\tilde{P}$ is still a Markov-chain rows $\Sigma = 1$

- Eigenvalues of P:

$$\lambda_i = (1-\gamma)\widetilde{\lambda_i} + (1-(1-\gamma))$$

- ≥ 0 if $\gamma \geq 1/2$

The lazy and time-reversable case: Convergence proof

$$\boxed{Q(x) \cdot p(X, Y) = Q(Y) \cdot p(Y, X)} \quad \Big| \; \frac{1}{\sqrt{Q(x)}} \cdot \frac{1}{\sqrt{Q(y)}}$$

$$\boxed{\sqrt{Q(x)} \cdot p(x, y) \cdot \frac{1}{\sqrt{Q(y)}} = \sqrt{Q(y)} \cdot p(y, x) \cdot \frac{1}{\sqrt{Q(x)}}}$$

$$\underbrace{\begin{pmatrix} \sqrt{Q(x)} & & \\ & \ddots & \\ & & \sqrt{Q(x)} \end{pmatrix} P \cdot \begin{pmatrix} \frac{1}{Q(Y)} & & \\ & \ddots & \\ & & \end{pmatrix}}_{A}$$

Symmetric matrix. A, P have same Eigenvalues

$A^T = A$

$v \cdot P^{(t)}$

$v^{(t)} = \mu_1^{(t)} \cdot u_1 + \dots + \mu_n \cdot u_n \rightarrow u_1$

$\mu_1 \cdot (\lambda_1)^t + \dots + \mu_n (\lambda_n)^t u_n$

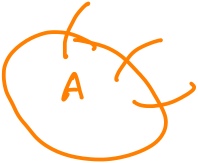
Conductance and Cheegers inequality

prob. of going from A to B.

Cut edges

- *Ergodic flow:*

$$Q(A,B) = \boxed{\sum_{X\in A, Y\in B} \boxed{\pi_s(X)p(X,Y)}}$$



$\pi_s(A)$

- Conductance:

$$\Phi = \underset{\substack{A\subseteq V\\ 0<\pi_s(A)\le 1/2}}{\overset{\text{min}}{\cancel{\max}}} Q(A,A^c)/\pi_s(A)$$

$\mu_1 u_1 + \mu_2 u_2 + \dots + \mu_n u_n$ A

$\mu_1 \cdot \mu_1 + \mu_2 \cdot \lambda_2^t u_2 + \dots$

$\downarrow$ exp. fast

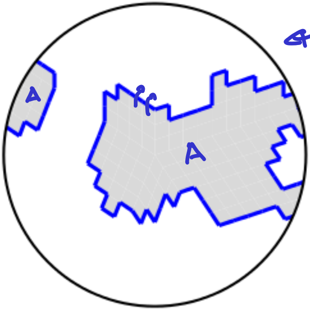
0

- If P is a reversible Markov chain, then

$$2\Phi \geq \boxed{1-\overset{\lambda_1}{}\lambda_2} \geq \boxed{\Phi^2/2} \qquad \frac{1}{p(n,1/\delta)}$$

- In our case: Rapid mixing if conductance is $\geq 1/poly(n,1/\delta)$

An isoperimetric inequality



Ball = K

$\frac{\mu(P')}{\mu(P)} \approx \delta$

$P(P';P') \geq \delta$

Applegate & Kannan

Isoperimetric inequality

$K \subseteq \mathbb{R}^n$ convex body and $\boxed{g(x) = e^{-\|x-\mu\|^2/2\sigma^2}}$ Gaussian. For any set $A \subset K$,

$$\int_{\partial_K A} g(x)\,dx \int_K g(x)\,dx \geq \boxed{\frac{\ln 2}{\sigma}} \int_A g(x)\,dx \int_{K\setminus A} g(x)\,dx.$$

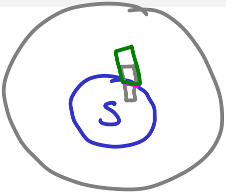
Small enough parallelepipeds

If $N \geq 3n/t_0$

- For two points $x, y \in \mathbb{R}^n$ in the same parallelopiped P *that intersects the unit ball,* *we have*
$$\frac{g(x)}{g(y)} \leq 2$$
- *For two neighboring parallelopipeds* P, P' that intersect the unit ball we have $\boxed{\frac{\delta}{2} \leq \frac{f(P)}{f(P')} \leq \frac{2}{\delta}.}$

Conductance of set S contained in unit ball

Q

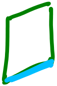


- S set of parallelepipeds in unit ball
- $A = \{(P, P') \colon P \in S, P' \in N(P) \setminus S\}$
- $\Phi(S) \boxed{\geq} \sum_{(P,P') \in A} Q(P) P(P, P') / Q(S)$
- $= \sum_{(P,P') \in A} Q(P') P(P', P) / Q(S)$
- $\approx \delta / n \cdot \sum_{(P,P') \in A} \boxed{Q(P')} / Q(S)$
- $\geq \delta^{\boxed{2}} / (nN) \dfrac{\int_{x \in \partial_{\mathscr{B}_n} S} g(x)\, dx}{\int_{x \in S} g(x)\, dx}$

Measure of Boundary

Measure of S.

- $\approx \delta^3 / n^{3.5}$
- Random walk is rapidly mixing



μ(Area) · height

δ/n

Main result

Theorem

There is a random edge pivot rule that solves a linear program using $poly(n, 1/\delta)$ *pivots in expectation.*

Independent on # of constraints !