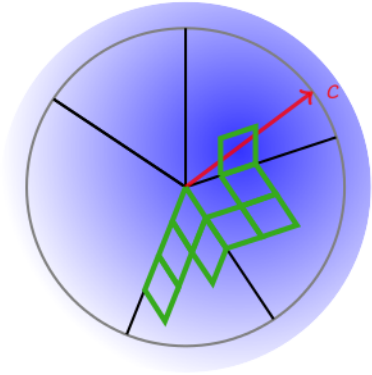
Randomized Algorithms for Linear programming



Friedrich Eisenbrand

November 18, 2014

Topics of these lectures

- Introduction to Linear Programming
- Simplex Algorithm
- Clarkson's algorithm
- Diameter: Bounds and open problems
- A simplex-algorithm guided by a random walk

Complexity of linear programing

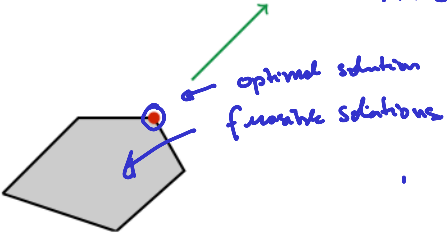
- Linear program:

$$\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$$

$c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$Ax \leq b$ → $a_1^T \cdot x \leq b_1$ ⋮ $a_m^T x \leq b_m$



- Important paradigm in mathematics, computer science, engineering ...

Complexity of linear programing

A, b, c integral

$M \geq |\text{Numbers}|$

- Linear program:

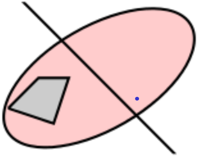
$$\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$$

$p(n, m, \log(M))$

- Solvable in *weakly polynomial time*

  (Khachiyan 79)

- Polynomial in *binary encoding length of input*



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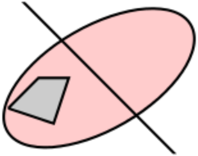
- Polynomial in *binary encoding length of input*
- Weakly polynomial running time natural for algorithms in *number theory* *(Mansour, et al. 91)*



$a \geq b \geq 2$

$\gcd(a,b) = \gcd(b,r)$

$a = q \cdot b + r$ Div. Rem.

$r \leq a/2$

Complexity of linear programing

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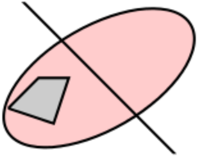

- Solvable in *weakly polynomial time*

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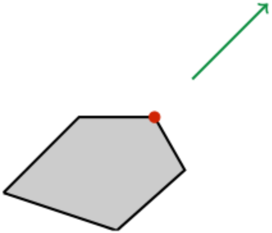
- Polynomial in *binary encoding length of input*
- Weakly polynomial running time natural for algorithms in *number theory* *(Mansour, et al. 91)*
- Seems *unnatural for combinatorial problems like linear programming*

Motivation

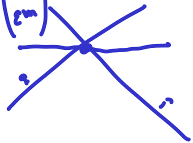
Problem

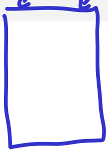
Can linear programming be solved in *strongly polynomial time*?



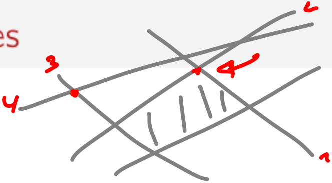
Assumptions

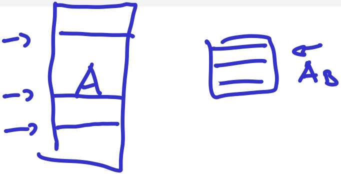
- $A \in \mathbb{R}^{m \times n}$ of full column rank.
- $Ax \leq b$ non-degenerate.





Bases and vertices





Bases

- $B \subseteq \{1, \dots, m\}$ is *basis* if A_B is invertible
- Basis B is feasible if unique solution x^* of

$$A_B x = b_B$$

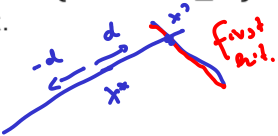
  satisfies $Ax \leqslant b$

- x^* is a *vextex* of polyhedron $\{x \in \mathbb{R}^n \colon Ax \leq b\}$.

improving

Theorem

If LP $\max\{c^T x : Ax \leq b\}$ has optimal solution, then there exists optimal vertex.



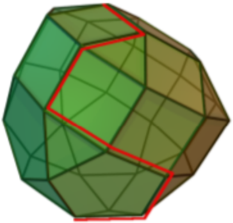
$A^{=} x \leq b^{=}$ ⇐ constraint tight at x^*

subsystem of $Ax \leq b$

$\operatorname{rank}(A^{=}) < n$ $\quad \exists\, d \neq 0 \quad A^{=} \cdot d = 0$

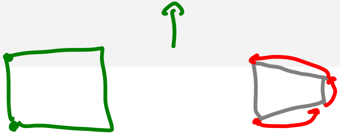
Simplex algorithm

- Walks from vertex to vertex of polyhedron
- First described by George Dantzig in 50's
- Is a *candidate* for strongly polynomial time algorithm



Simplex algorithm

- Start with feasible basis.
- While x^* not optimal ($\lambda^T A_B = c$ but $\lambda_i < 0$)
- Follow ray $x^* + \mu d$ with $Ad = -e_i$) until new inequality j becomes tight
- Set $B := B - i + j$



Theorem

If LP satisfies assumtions, then simplex terminates with either

1. assertion of unboundedness
2. optimal solution.

Proof:
- Improvement each step
- Number of bases is finite $\binom{m}{n}$ exp inn.

Total unimodularity

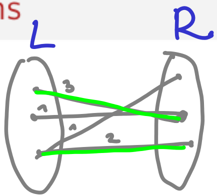
Definition

$A \in \{\pm 1, 0\}^{m \times n}$ is *totally unimodular if each $k \times k$ sub-determinant is* $0, \pm 1$

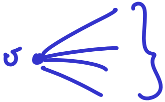
Theorem

If A is totally unimodular and b is integral, then each vertex of $\{x \colon Ax \leqslant b\}$ is integral.

Example: Matchings in bipartite graphs



- $G = (A \cup B, E)$ bipartite
- $M \subset E$ *matching* of *no two edges in M share an endpoint*
- $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise.} \end{cases}$
-



$$\max \sum_{e \in E} w_e x_e$$
$$\forall v \in V : \sum_{e \in \delta(v)} x_e \leqslant 1$$
$$x \geq 0$$

- *Each vertex is* $0/1$*-vector.*

Clarkson's algorithm

max $c^T \cdot x$

$Ax \leq b$ ← m constraints

- Randomized algorithm
- Reduces in expected polynomial time LP with m constraints to $O(m)$ LPs with $O(n^2)$ constraints

Consequence

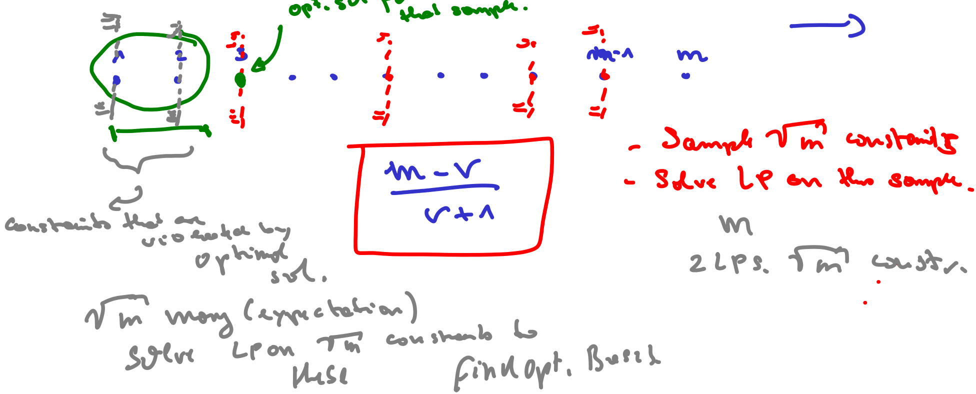
Randomized algorithm with expected strongly polynomial running time exists if and only such an algorithm exists for LPs with $O(n^2)$ constraints.

if

Quiz

- $H = \{1, \ldots, m\}$, $r \in H$, $R \in \binom{H}{r}$ drawn uniformly at random
- $V_R = \min\{i \in R\} - 1$
- *What is* $E[|V_R|]$ *?*

$r = 4$



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- *What is* $E[|V_R|]$ *?*

Answer. $(m - r)/(r + 1)$

Proof

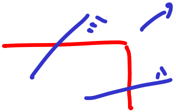
- For $Q \subseteq H$ and $j \in H$ define $\chi(Q, j) = \begin{cases} 1 & \text{if } j < \min\{i \in Q\}, \\ 0 & \text{otherwise.} \end{cases}$
- $E[|V_R|] = \left(\sum_{R \in \binom{H}{r}} \sum_{j \in H \setminus R} \chi(R, j) \right) / \binom{m}{r}$
- One has $\binom{m}{r} \cdot (m - r) = \binom{m}{r+1} \cdot (r + 1)$

max $c^T \cdot x$

$c \geq 0$

Framework

- Given: Set H of m linear constraints in $\mathbb{R}^n$ and $H^- = \{x(i) \leqslant M \mid i = 1, \ldots, n\}$ explicit upper bounds
- For $G \subseteq H$, $x^*(G)$ is *unique max.* point satisfying all $h \in G \cup H^-$
- Task: *Compute $x^*(H)$*



Basis

$B \subseteq G$ is called *Basis* of G, if $x^*(B) = x^*(G)$ and for each $b \in B$ one has $x^*(B - b) > x^*(B)$.

identify all violated constr. V

Lemma

Let B be a basis of H and let $G \subseteq H$. One has $x^*(G) > x^*(H)$ if and only if there exists $b \in B$ with $x^*(G)$ violates b.

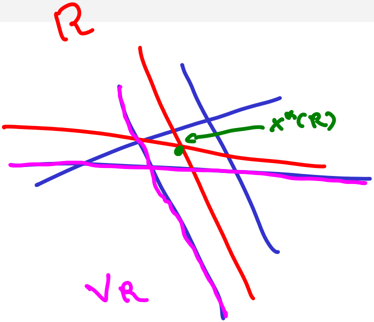
then we know that one basis element not in G is in V

Quiz

H: set of constraints

- Choose $R \in \binom{H}{r}$ uniformly at random
- $V_R = \{h \in H \mid x^*(R) \text{ violates } h\}$
- *What is* $E[|V_R|]$?

$$\frac{m-r}{(r+1)} \cdot n$$



Quiz

- Choose $R \in \binom{H}{r}$ uniformly at random
- $V_R = \{h \in H \mid x^*(R) \text{ violates } h\}$
- *What is* $E[|V_R|]$?

Answer. at most $((m - r)/(r + 1)) \cdot n$

H: Set of constr., $R \in \binom{H}{r}$. $V_R = \{ h \in H : x^*(R) \text{ violates } h \}$.

$$\binom{H}{r} \cdot E[|V_R|] =$$

$$\chi(Q, h) = \begin{cases} 1 & \text{if } x^*(Q) \text{ violates } h \\ 0 & \text{otherwise.} \end{cases}$$

$$= \sum_{R \in \binom{H}{r}} \sum_{h \in H \setminus R} \chi(R, h)$$

again $\binom{H}{r} \cdot (m-r)$
$= \binom{H}{r+1} \cdot (r+1)$

$$= \sum_{Q \in \binom{H}{r+1}} \sum_{h \in Q} \chi(Q-h, h) \leq n$$

$$\Rightarrow E[|V_R|] \leq \binom{m}{r+1} / \binom{m}{r} \cdot n = \frac{m-r}{r+1} \cdot n$$

Proof

- $E[|V_R|] = \left(\sum_{R\in\binom{H}{r}} |V_R|\right) / \binom{m}{r}$
- For $Q \subseteq H$ and $h \in H$ define $\chi(Q,h) = \begin{cases} 1 & \text{if } x^*(Q) \text{ violates } h, \\ 0 & \text{otherwise.} \end{cases}$

$$\begin{aligned}
\binom{m}{r} E(|V_R|) &= \sum_{R\in\binom{H}{r}} \sum_{h\in H\setminus R} \chi(R,h) \\
&= \sum_{Q\in\binom{H}{r+1}} \sum_{h\in Q} \chi(Q-h,h) \\
&\leqslant \sum_{Q\in\binom{H}{r+1}} d \\
&= \binom{m}{r+1} \cdot n.
\end{aligned}$$

Sampling Lemma

Lemma

Let G and H (multi-)sets of constraints $|H| = m$ and let $1 \leqslant r \leqslant m$. Then for random $R \in \binom{H}{r}$:

$$E[|V_R|] \leqslant n(m-r)/(r+1),$$

where $V_R = \{h \in H \mid x^*(G \cup R) \text{ violates } h\}$.

Sampling Lemma

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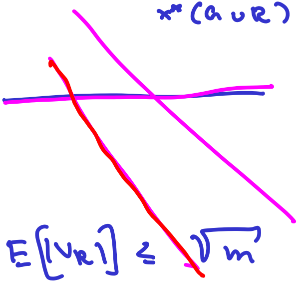
where $V_R = \{h \in H \mid x^*(G \cup R) \text{ violates } h\}$.

Set $r = \lceil n \cdot \sqrt{m} \rceil$ then

$$E[|V_R|] \leqslant n \cdot (m-r)/(r+1) \leqslant n \cdot m/r \leqslant \sqrt{m}.$$

Clarkson's algorithm I

1. Input: H with $|H| = m$
2. $r \leftarrow n\sqrt{m}$ Sample size
3. $G \leftarrow \emptyset$ Constraints we collect.
4. REPEAT (expect $O(n)$ iterations)
   - 4.1 Choose random $R \in \binom{H}{r}$
   - 4.2 Compute $x^* = x^*(G \cup R)$
   - 4.3 $V_R \leftarrow \{h \in H \mid x^* \text{ violates } h\}$
   - 4.4 IF $|V_R| \leqslant 2\sqrt{m}$ happens with Prob $\geq 1/2$
     THEN $G \leftarrow G \cup V_R$
5. UNTIL $V_R = \emptyset$



$E[|V_R|] \leq \sqrt{m}$

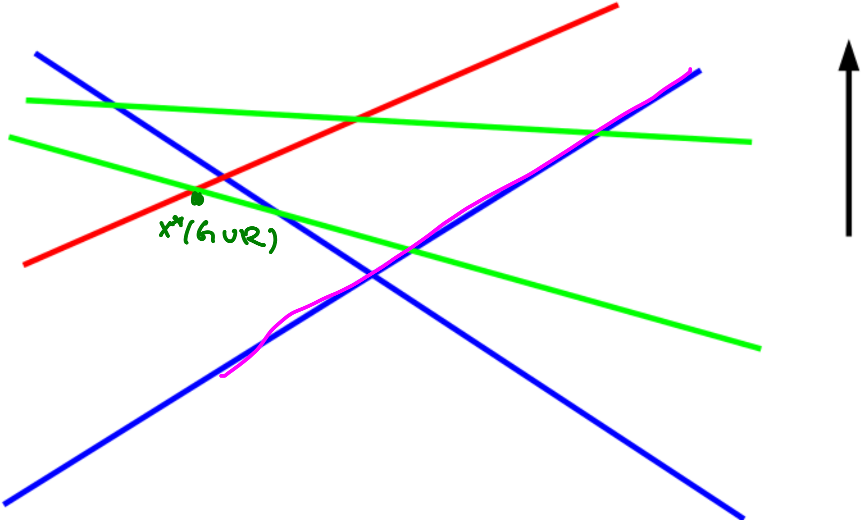
Successful iteration.

# of successful it. $\leq n$

Invariant: G contains at most $2 \cdot n \cdot \sqrt{m}$ constraints

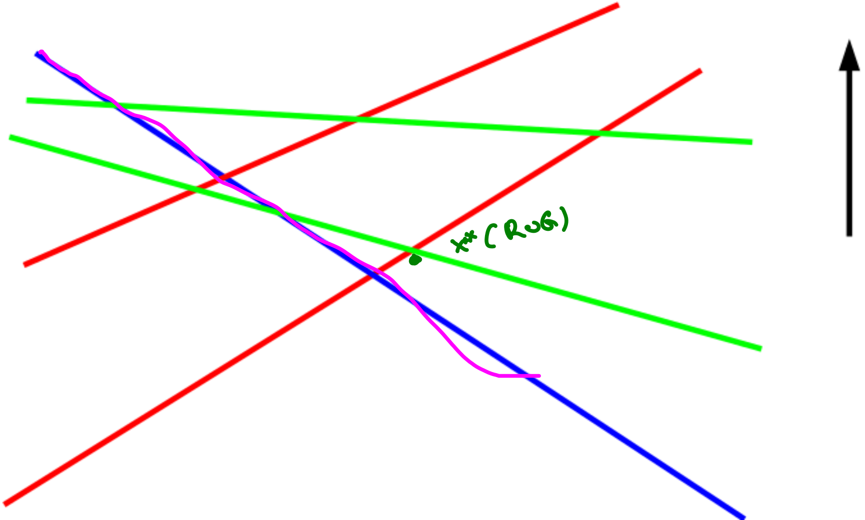
Example

R, G, B



Example

R, G, B



Analysis

In Step (4.c): $E[|V|] \leqslant \sqrt{m}$.
Let B be optimal basis.

- Each successful iteration, a *new* element of B enters G
- Thus at most n succ. it.
- $P(|V_R| > 2\sqrt{m}) \leqslant 1/2$ Markow inequality
- Expected number of iterations is $2n$

Clarkson 1 performs:

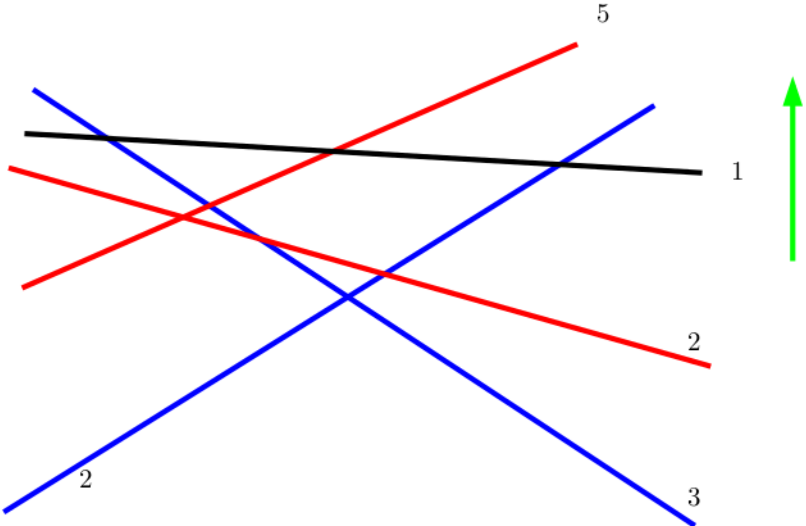
- Expected $2\,n$ calls to linear programming *oracle* with at most $3 \cdot n\sqrt{m}$ constraints
- Expected number of $O(n^2 \cdot m)$ *arithmetic operations*

Clarkson's algorithm II

- Each $h \in H$ is assigned a *multiplicity* μ_h.
- In the beginning $\mu_h = 1$ for all $h \in H$.
- Sample size r is small
- Idea: If $x^*(R)$ violates h, then *multiplicity/probability* is doubled
- Constraints of optimum basis become much more likely to be drawn next time
- We stop if R contains optimum basis

Example

R, B



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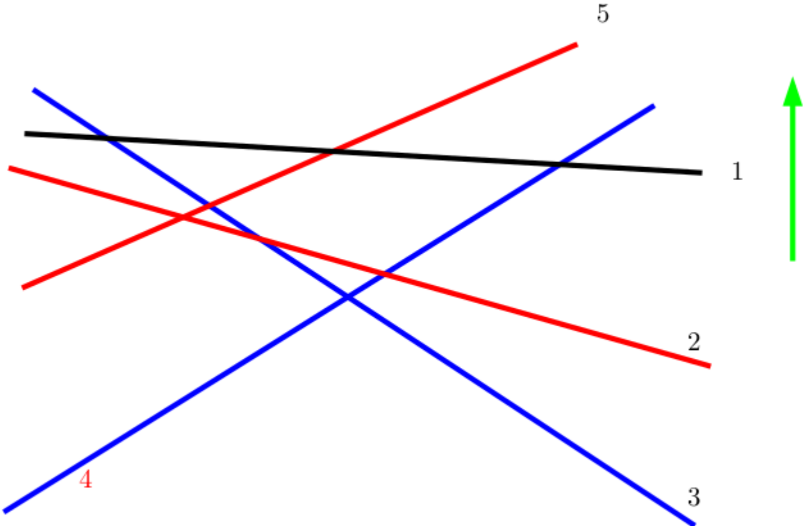
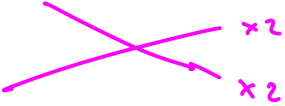


Clarkson 2

1. INPUT: H, $|H| = m$
2. $r \leftarrow 6 \cdot n^2$
3. REPEAT:
   - 3.1 Choose random $R \in \binom{H}{r}$
   - 3.2 Compute $x^* = x^*(R)$
   - 3.3 $V_R \leftarrow \{h \in H \mid x^* \text{ violates } h\}$
   - 3.4 IF $\mu(V_R) \leqslant 1/(3d)\mu(H)$ THEN for all $h \in V$ do $\mu_h \leftarrow 2\,\mu_h$
4. UNTIL $V_R = \emptyset$



Lemma

B optimal basis, after kn *successful iterations (entering re-weighting step):*

$$2^k \leqslant \mu(B) \leqslant m\, e^{k/3}, \text{ for basis } B \text{ of } H.$$

$1+x \leq e^x$

$2^k > m \cdot e^{k/3}$

$k > \log(m) + c.h.$

$k = O(\log m)$ iterations.

each success it. one element of B is doubled.

$\mu_{new}(H) \leq \left(1+\frac{1}{3\cdot n}\right) \cdot \mu_{old}(H)$

$\mu(B^i) \leq \left(1+\frac{1}{3\cdot n}\right)^i \cdot m \leq e^{\frac{i}{3\cdot n}} \cdot m \leq e^{\frac{k}{3}} \cdot m$

Complexity Clarkson 2

- $2^k \leqslant me^{k/3}$ implies $k \in O(\log m)$
- Expected number of $O(n \cdot \log m)$ iterations

Clarkson 2 requires

- expected number of $O(n^2 m \log m)$ *arithmetic operations*
- expected $6n \ln m$ *base cases* with $6 \cdot n^2$ constraints

Combining Clarkson 1 and 2

- $O(n^2 \cdot m)$ arithmetic operations
- $2 \cdot n$ calls to Clarkson 2 on $O(n\sqrt{m})$ constraints
  - $O(n^2\sqrt{m}\log m)$ arithmetic operations
  - $O(n \log m)$ calls to LP-oracle with $6 \cdot n^2$ constraints

Linear program can be solved

- with expected $O(n^3 \cdot m)$ *arithmetic operations*
- and $O(n^2 \cdot \log m)$ *oracle calls* to solve an LP with $6 \cdot n^2$ constraints
- in *linear time* if n is fixed

(Clarkson 1995)

The polynomial Hirsch conjecture

Polynomial Hirsch conjecture

Is $\Delta(n, m)$ bounded by a polynomial in n and m?

The polynomial Hirsch conjecture

Polynomial Hirsch conjecture

Is $\Delta(n,m)$ bounded by a polynomial in n and m?

Classical Hirsch conjecture

... (polytopes)

$$\Delta(n,m) \leqslant m-n$$

was refuted by Santos (2010).

What is known?

What is known?

- Best known bound:

$$\Delta(n, m) \leqslant m^{1+\log n}$$

(Kalai, Kleitman 1992)

$$\Delta(n, m) \leqslant (m - n)^{\log n}$$

(Todd 2014)

What is known?

- Best known bound:

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(Todd 2014)

- Huge gap between (linear) lower bound and best known upper bound

This talk . . .

. . . is around some recent developments on this question.

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- Best known bounds in light of a simple abstraction
  - $m^{\log n+1}$ (Kalai and Kleitman 1992)
  - $O(m)$ for fixed n (Larman 1970)
- Almost quadratic lower bound for abstraction

  (E., Hähnle, Razborov and Rothvoß 2010)
- Upper bound in dimension and $1/\delta$

  (Bonifas, Di Summa, E., Hähnle and Niemeier 2012)
- Random edge solves linear programs in time $poly(n, 1/\delta)$

  (E. & Vempala 2014)

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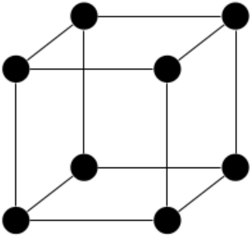
. . . and *open problems around these results.*

In light of a simple abstraction . . .

Base Abstraction

Base abstraction is graph $G = (V, E)$ with $V \subseteq \binom{[m]}{n}$ such that

- every pair $u, v \in V$ is connected by a path in G whose vertices all contain $u \cap v$.



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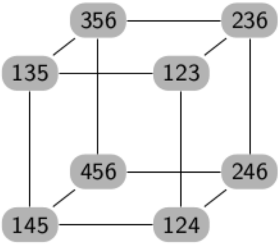

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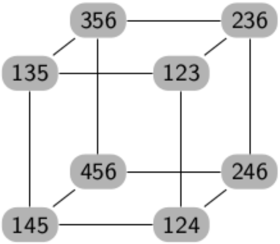

Kalai & Kleitman (1992)

$D(n, m) \leqslant m^{1+\log n}$

Connected Layer Families

- Partition V into *layers* $\mathcal{L}_1, \ldots, \mathcal{L}_\ell$ such that
  - every set of symbols that is covered on layers i and j, $i < j$, is also covered on each layer in between.
- Such a partition is a *connected layer family*, ℓ is its *height*.
- From base abstraction: $\mathcal{L}_i$ are vertices at distance i from s.

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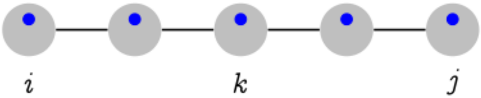


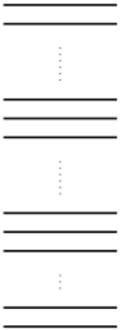
Upper Bound: Kalai & Kleitman

Theorem

$h(n, m) \leqslant m^{1+\log n}$

Proof.

We have $h(n, m) \leqslant 2h(n, \lfloor m/2 \rfloor) + h(n-1, m-1)$.



$\square$

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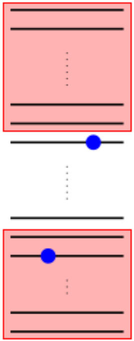

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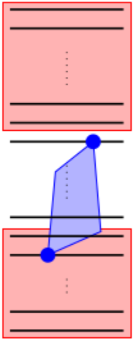

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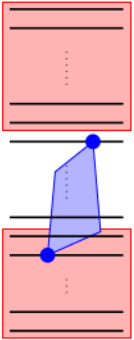

Proof.

We have $h(n, m) \leqslant 2h(n, \lfloor m/2 \rfloor) + h(n-1, m-1)$.
Solve the recurrence by induction on n and m:

$$\begin{aligned} h(n,m) &\leqslant 2h(n, \lfloor m/2 \rfloor) + h(n-1, m-1) \\ &\leqslant 2\sum_{i=2}^{n} h(i, \lfloor m/2 \rfloor) + h(1, m) \\ &\leqslant 2(n-1)(2n)^{\log m - 1} + m \\ &\leqslant (2n)^{\log m} \end{aligned}$$

□

Linear bound in fixed dimension

- If n is fixed, then diameter is linear in m.

(Larman 1970, Barnette 1974)

Theorem
$D(n, m) \leqslant 2^{n-1} \cdot m - 1.$

Lower bound

Theorem (E., Hähnle, Rothvoß & Razborov 2010)

$D(n, O(n)) = \Omega(n^2/\log n)$.

Lower bound

Theorem (E., Hähnle, Rothvoß & Razborov 2010)

$D(n, O(n)) = \Omega(n^2/\log n)$.

Theorem (Santos 2013)

Diameter of pure simplicial complexes can be exponential.

Hähnle's Conjecture

Hähnle's Conjecture: See discussion on Polymath 3

Is $h'(d,n) = d(n-1)+1$?

$d =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n = 1$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
4	$\surd$	$\surd$	+	+	+	+	+	+	+	+	+	+	+		
5	$\surd$	$\surd$	+	+	+	+	+								
6	$\surd$	$\surd$	+	+	+										
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8	$\surd$	$\surd$	+												
9	$\surd$	$\surd$													

A polynomial bound in n and $1/\delta$

$\max\{c^T x \colon Ax \leqslant b\}$

- Suppose each row a_i of A satisfies $||a_i|| = 1$
- Distance of row to subspace generated by other rows is $\geq \delta$
- *δ-distance property*

(Brunsch & Röglin 2013)

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δ-distance property: Motivation

- P flow polytope: quadratic upper bound. (Orlin 1997)

- *Transportation polytope: linear upper bound. (Brightwell, v.d.Heuvel and Stougie 2006)*
- $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ with A *totally unimodular: polynomial upper bound*

  $O(m^{16}n^3\log(mn)^3)$ *(Dyer and Frieze 1994)*

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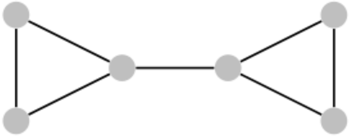
Totally unimodular matrices

... satisfy the δ-distance property with $\delta = 1/n$.

More than totally unimodular matrices

- *Edge-node incidence matrix of graphs*
- *Largest* minor: exponential in $|V|$
- $\delta = \Omega(1/\sqrt{V})$



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

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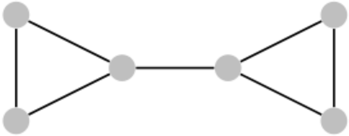


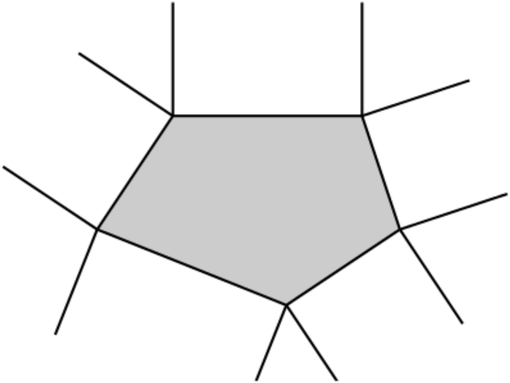
- Rows do not need to be integral or rational
- *Geometric property*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Upper bound: Proof method

- Assume $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ is non-degenerate
- Associate a *volume* to each vertex
- Estimate number of *Breadth-First-Search* iterations until sum of volumes of visited vertices exceeds half of the total volume

Pivoting and normal cones



- Normal cones do not intersect in interior.
- Two vertices are neighbors if and only if their normal cones share a facet.

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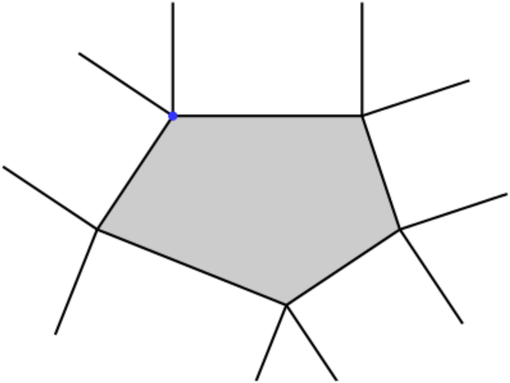

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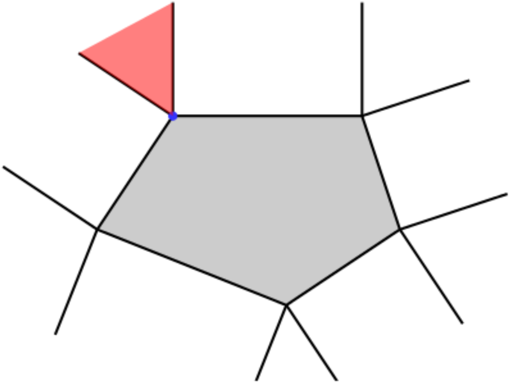

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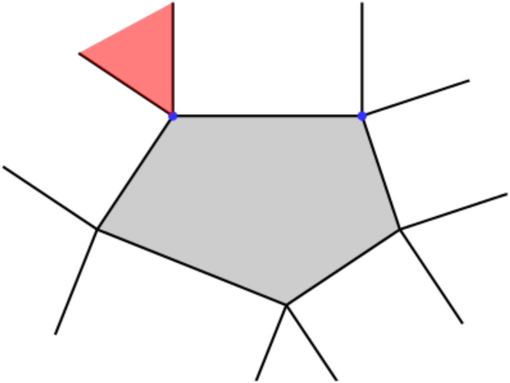

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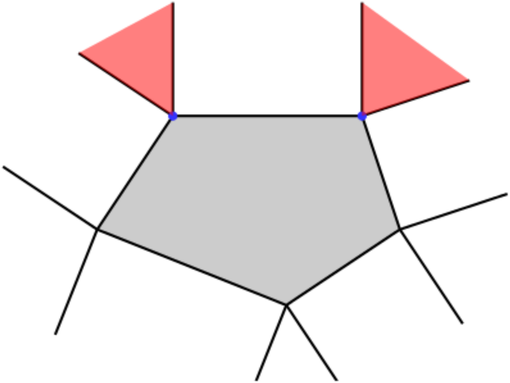

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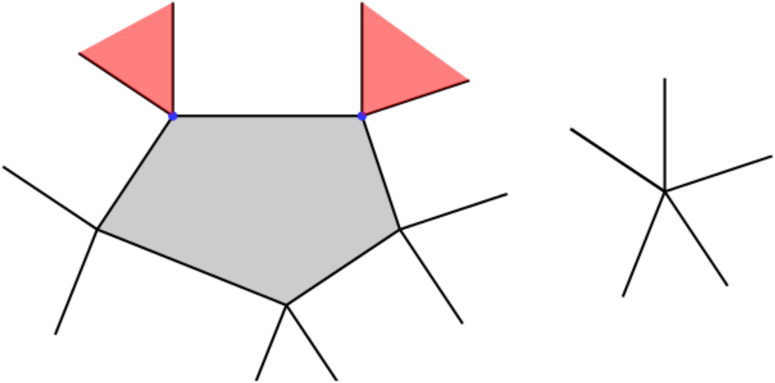

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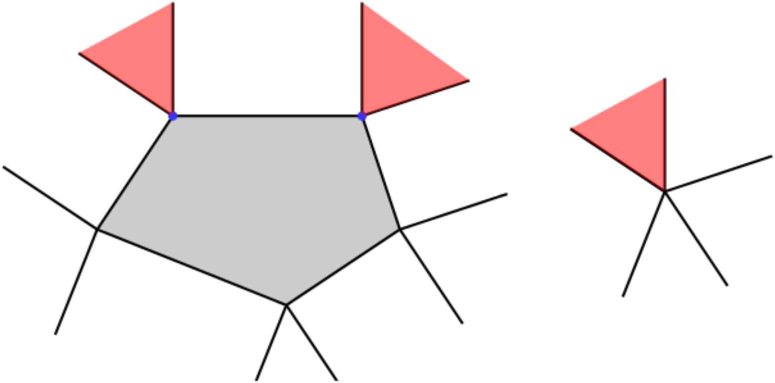

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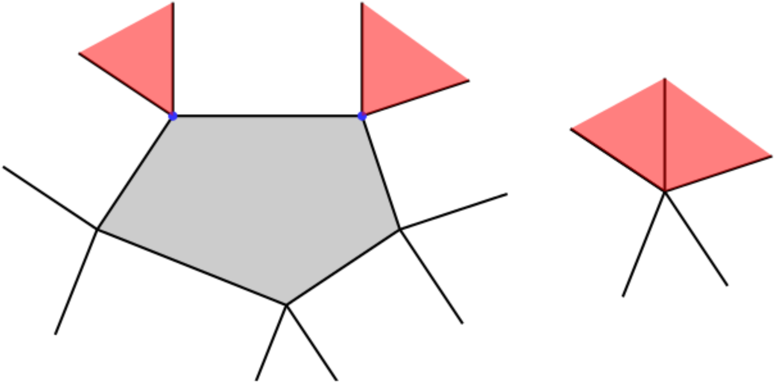

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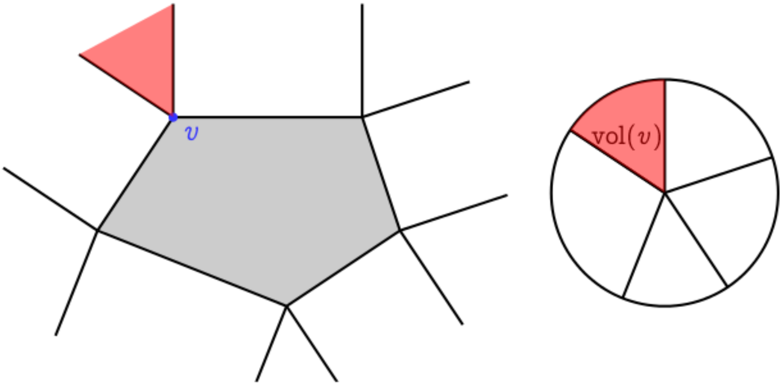
Volume of a vertex



- $\text{vol}(v) = \mathscr{B}_n \cap C_v$

Volume expansion

Lemma

Let $I \subseteq V$ with $\mathrm{vol}(I) \leqslant (1/2) \cdot \mathrm{vol}(B_n)$.
Volume of neighborhood of I is at least

$$\mathrm{vol}(\mathcal{N}(I)) \geqslant \sqrt{\frac{2}{\pi}}(\delta/n^{1.5}) \cdot \mathrm{vol}(I).$$

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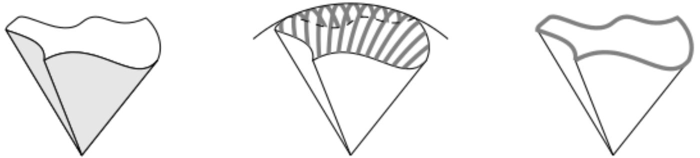
Theorem

Diameter is bounded by $O(n^{2.5}/\delta \cdot \ln(n/\delta))$.

Proving the volume expansion lemma

- $I \subseteq V$, $S = \bigcup_{v \in I} (C_v \cap B_n)$ *spherical cone*



(a) Dockable surface of S.

(b) Base of S.

(c) Relative boundary of the base of S.

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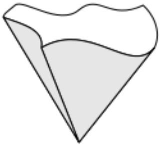

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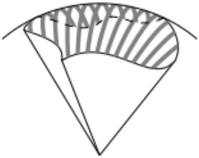

(e) Base of S.

(f) Relative boundary of the base of S.

Expansion lemma follows from inequalities

$$\frac{D(S_v)}{\mathrm{vol}(S_v)} \leqslant n^2/\delta \quad \text{and} \quad \frac{D(S)}{\mathrm{vol}(S)} \geqslant \sqrt{\frac{2n}{\pi}}.$$

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$$\sum_{v \in \mathscr{N}(I)} D(S_v) \geqslant D(S)$$

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$$\mathrm{vol}(\mathscr{N}(I))n^2/\delta \geq \sum_{v \in \mathscr{N}(I)} D(S_v) \geqslant D(S) \geq \mathrm{vol}(S)\sqrt{\frac{2\pi}{n}}$$

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$$\mathrm{vol}(\mathscr{N}(I)) \geq \sqrt{2\pi} \cdot \delta/n^{1.5} \cdot \mathrm{vol}(S)$$

The inequalities $\frac{D(S_v)}{\mathrm{vol}(S_v)} \leqslant n^2/\delta$ and $\frac{D(S)}{\mathrm{vol}(S)} \geqslant \sqrt{\frac{2n}{\pi}}$.

- First one is immediate from δ-distance property

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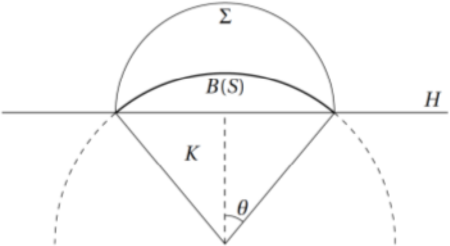


- Second inequality
  1. *Isoperimetric inequality*
  2. *Worst case cone is generated by shperical cap*
  3. *Worst case spherical cap is half-ball*

Result

Theorem (Bonifas et al. 2012)

The diameter is bounded by $O(n^{2.5}/\delta \cdot \ln(n/\delta))$.

Algorithmic results

- Dyer and Frieze (1994): polynomial randomized simplex algorithm for TU-LPs
- Brunsch and Röglin (2013): Shadow vertex finds short path between *designated vertices in expected polytime (in n, m and $1/\delta$)*
- *E. & Vempala (2014): Random-edge variant of simplex algorithm solves LPs in expected polytime (in n, m and $1/\delta$).*

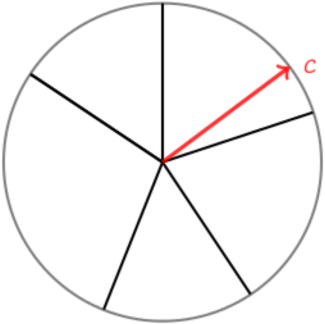
Expected number of pivots polynomial in n and $1/\delta$ only

Walking in the space of cones

If we

- start within cone of feasible solution
- leave a cone only through facet
- do not cross cones in one step

then we can keep track of optimal basis.



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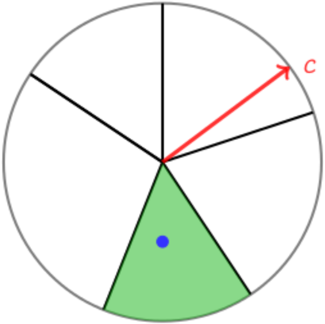

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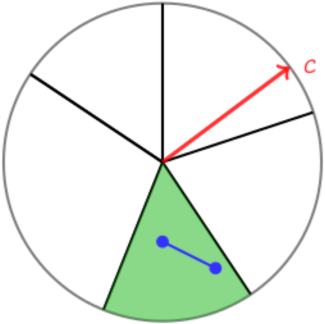

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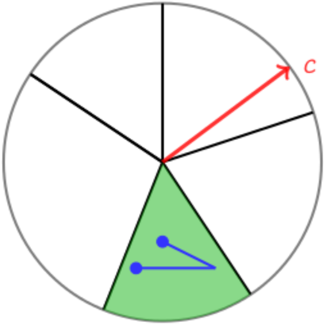

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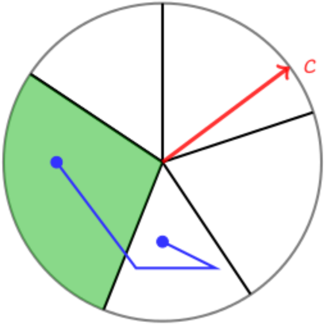

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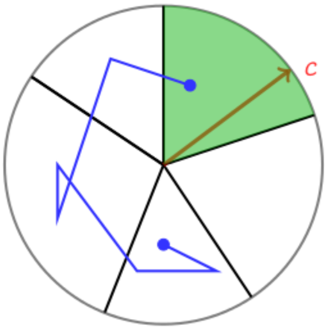

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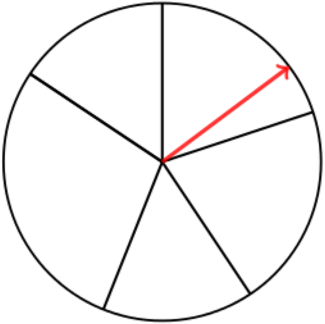
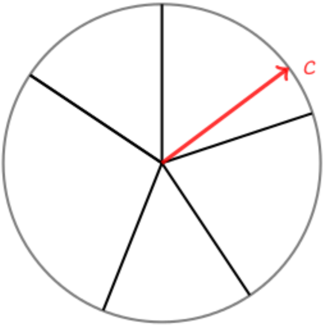


A random walk (Sketch!)

- Partition space of cones into small parallelepipeds, as in (Dyer and Frieze 1994)



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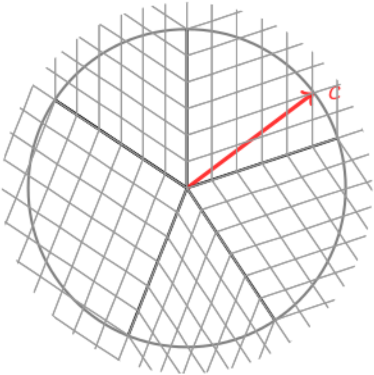

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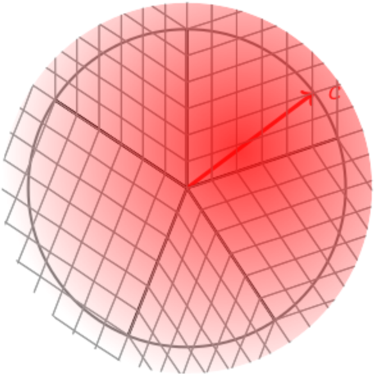

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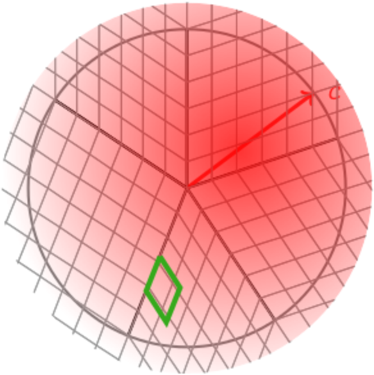

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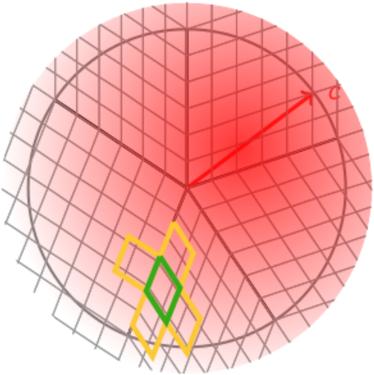
- Lazy, time-reversable Markov chain with *stationary distribution* *proportional to* measure of parallelepipeds



A random walk (Sketch!)

- Partition space of cones into small parallelepipeds, as in (Dyer and Frieze 1994)
- Consider Gaussian $g(x) = \exp(-||x - c/8||^2/(2t_0))$
- At given parallelepiped P:
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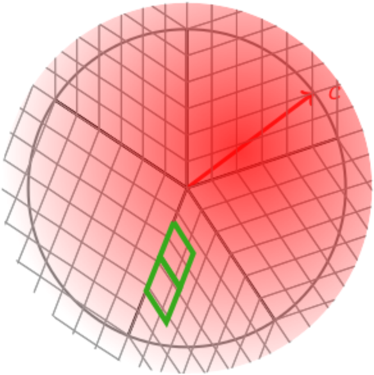

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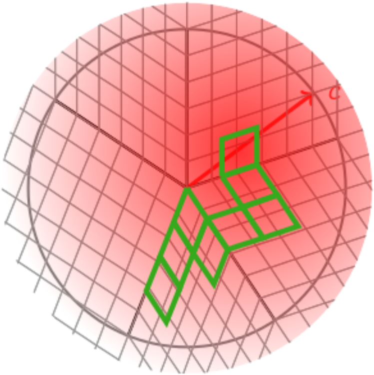

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- Is simplex algorithm generic machinery for efficient combinatorial algorithms (Matchings, Submodular functions, ...)?