Bisimulation for Lattice-Valued Doubly Labeled Transition Systems

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Formal models: specification of reactive systems

- Transition Systems
- Labeled Transition Systems
- Kripke Structures
- ...

Bisimulation: behavior equivalence.

An assumption: The systems are complete and consistent.

Classical (two-valued) transition systems and classical (two-valued) bisimulation work well.

Unfortunately, the available information is incomplete and impreciseness in complex systems.

- Cyber-physical Systems
- Internet of Things
- ...

How to represent these systems?...

Evolution of Formal Models

- Nondeterminism
- Probabilistic
- Fuzz
- Lattice-valued
- ...

Evolution of Bismulation

- λ-bisimulation
- (η, δ)-bisimilarity
- Bisimulation for deterministic and nondeterministic fuzzy systems
- Bisimulations (forward, backward, forward-backward and backward-forward) for fuzzy automata
- Lattice-valued simulation based on lattice-valued Kripke structure (latticed-valued structure is a De Morgan algebra)
- ...

We focus on ...

- Our Formal Model
  - Extend Doubly transition systems to Lattice-valued doubly transition systems
  - Residuated lattices as the structure of truth values of transition system
- Our Bisimulation
  - Extend approximate Bisimulation to lattice-valued bisimulation
  - Lattice-valued version of (η, δ)-bisimulation

Residuated Lattices

Definition

A *residuated lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ in which $\wedge, \vee, \otimes$, and $\to$ are binary operators on the set L and

(1) $(L, \wedge, \vee)$ is a bounded lattice with 0 as smallest and 1 as greatest element,

(2) $\otimes$ is commutative and associative, with 1 as neutral element, and

(3) $x \otimes y \leq z$ iff $x \leq y \to z$ for all x, y and z in L (residuation principle).

We will use the notation $\neg x$ for $x \to 0$ (*negation*), $x \leftrightarrow y$ for $(x \to y) \wedge (y \to x)$.

Strong Residuated Lattices

Definition

If $\mathcal{L}$ is a residuated lattice, and satisfies,

$$a \to b \vee c = (a \to b) \vee (a \to c)$$

for any $a, b, c \in L$, then $\mathcal{L}$ is called a *normal* residuated lattice. If a residuated lattice $\mathcal{L}$ satisfies , $\neg\neg a = a$ for any $a \in L$, then $\mathcal{L}$ is called a *regular* residuated lattice.

If a residuated lattice is both normal and regular, we refer it as *strong residuated lattice*. We use the notations $\mathcal{SRL}$ to denote the class of strong residuated lattices.

Lattice-valued Sets

Definition

- A *lattice-valued set* (for short, L-set) A on a universe U is a mapping from U to L. The set of all lattice-valued sets on universe U is denoted as $\mathcal{L}(U)$.
- Let A, B be non-empty sets. A *lattice-valued relation* (for short, L-relation) on A and B is any mapping from $A \times B$ into L, that is to say, any L-subset of $A \times B$.

L-equivalence Relations

Definition

A *L-equivalence relation* θ on a set A is a mapping $\theta : A \times A \to L$ satisfying

(1) reflexive:$\theta(x, x) = 1$, for every $x \in A$;

(2) symmetric: if $\theta(x, y) = \theta(y, x)$, for all $x, y \in A$;

(3) transitive: if for all $x, y, z \in A, \theta(x, y) \wedge \theta(y, z) \leq \theta(x, y)$.

LDLTS

Definition

A *lattice-valued doubly labeled transition system* (for short, *LDLTS*), $\mathcal{M}$, is defined as a tuple $(\mathcal{L}, S, S^0, \Sigma, AP, R, \mathcal{V}, \theta)$, where

(1) $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a residuated lattice;

(2) S is a *finite* set of states;

(3) S^0 is an L-set of initial states;

(4) Σ is a *finite* set of action labels;

(5) AP is a *finite*, non-empty set of atomic propositions;

(6) $R : S \times \Sigma \times S \rightarrow L$ is an L-transition relation on S;

(7) $\mathcal{V}$ is a valuation function $\mathcal{V} : S \times AP \rightarrow L$, assigning a truth value in L to each atomic proposition in every state; and

(8) θ is an L-equivalence relation on Σ.

LDLTS v.s DLTS

The LDLTS differs from a standard DLTS:

- the initial state set and transition relation are both L-set,
- proposition valuation function maps a state to a mapping from propositions to element of L, and
- the set of actions Σ is equipped with an L-equivalence relation.

In what follows, unless specially noted, we consider the fixed LDLTSs $\mathcal{M}_i = (\mathcal{L}, S_i, S_i^0, \Sigma, AP, R_i, \mathcal{V}_i, \theta), i = 1, 2$, and assume that all residuated lattices under consideration are *finite*.

L-bisimulation

Definition

Given two LDLTSs $\mathcal{M}_i$, $i = 1, 2$. An L-relation, $\mathcal{R} \in \mathcal{L}(S_1 \times S_2)$, is a lattice-valued bisimulation (L-bisimulation) between M_1 and M_2, if for all $s_1 \in S_1, s_2 \in S_2$,

$$\mathcal{R}(s_1, s_2) = \mathcal{R}_{AP}(s_1, s_2) \wedge \mathcal{R}_t(s_1, s_2)$$
$$\wedge \bigwedge_{a \in \Sigma} \bigwedge_{s_2' \in S_2} (R_2(s_2, a, s_2') \rightarrow \bigvee_{b \in \Sigma} \bigvee_{s_1' \in S_1} (R_1(s_1, b, s_1') \otimes \theta(a, b) \otimes \mathcal{R}(s_1', s_2')))$$

where

$$\mathcal{R}_{AP}(s_1, s_2) = \bigwedge_{p \in AP} (\mathcal{V}_1(s_1, p) \leftrightarrow \mathcal{V}_2(s_2, p))$$
$$\mathcal{R}_t(s_1, s_2) = \bigwedge_{a \in \Sigma} \bigwedge_{s_1' \in S_1} (R_1(s_1, a, s_1') \rightarrow \bigvee_{b \in \Sigma} \bigvee_{s_2' \in S_2} (R_2(s_2, b, s_2')$$
$$\otimes \theta(a, b) \otimes \mathcal{R}(s_1', s_2')))$$

L-bisimilarity

L-bisimilarity, in symbols $\sim_L$, is defined as

$$\sim_L = \bigcup \{\mathcal{R} \mid \mathcal{R} \text{ is an L-bisimulation}\}.$$

search for the greatest L-bisimulation relation

Algorithm 1: For two LDLTS $\mathcal{M}_i, i = 1, 2$, define the following sequence $\{\mathcal{R}_i \mid i \in \mathbb{N}\}$ of L-subsets of $\mathcal{L}(S_1 \times S_2)$: for each state $s_1 \in S_1, s_2 \in S_2$,

$$\mathcal{R}_0(s_1, s_2) = \bigwedge_{p \in AP} (\mathcal{V}_1(s_1, p) \leftrightarrow \mathcal{V}_2(s_2, p))$$

$$\mathcal{R}_{i+1}(s_1, s_2) = \mathcal{R}_i(s_1, s_2)$$

$$\wedge \bigwedge_{a \in \Sigma} \bigwedge_{s_1' \in S_1} (R_1(s_1, a, s_1') \to \bigvee_{b \in \Sigma} \bigvee_{s_2' \in S_2} (R_2(s_2, b, s_2') \otimes \theta(a, b) \otimes \mathcal{R}_i(s_1', s_2')))$$

$$\wedge \bigwedge_{a \in \Sigma} \bigwedge_{s_2' \in S_2} (R_2(s_2, a, s_2') \to \bigvee_{b \in \Sigma} \bigvee_{s_1' \in S_1} (R_1(s_1, b, s_1') \otimes \theta(a, b) \otimes \mathcal{R}_i(s_1', s_2'))).$$

Theorem

Let $\{\mathcal{R}_{i\in\mathbf{N}}\}$ be the sequences of L-relation sets defined by Algorithm 1 and $\sim_L$ be the L-bisimilarity relation between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then, the following properties hold: for all $i \geq 0$,

(1) $\mathcal{R}_{i+1} \subseteq \mathcal{R}_i$;

(2) $\bigcap_{i\in\mathbf{N}} \mathcal{R}_i$ *is an L-bisimulation relation;*

(3) $\bigcap_{i\in\mathbf{N}} \mathcal{R}_i$ *is the greatest L-bisimulation relation;*

(4) $\sim_L$ *is the greatest L-bisimulation relation, i.e.* $\sim_L = \bigcap_{i\in\mathbf{N}} \mathcal{R}_i$.

Composition of LDLTS

Definition

Let $\mathcal{M}_i, i = 1, 2$, be two LDL TSs and $AP_1 \cap AP_2 = \emptyset$. The approximate synchronization operator $\|_\beta$, $\beta \in L$, acting on the two systems results in another transition system is the LDLTS

$$\mathcal{M}_1 \|_\beta \mathcal{M}_2 = (L, S_1 \times S_2, S_1^0 \times_{\mathcal{L}} S_2^0, \Sigma \times \Sigma, R, AP_1 \cup AP_2, \mathcal{V}, \theta')$$

Note that the composite LDLTS $\mathcal{M} = \mathcal{M}_1 \|_\beta \mathcal{M}_2$ is quite different from the LDLTSs $\mathcal{M}_1, \mathcal{M}_2$, in the following sense:

- The set of actions of $\mathcal{M}$ is also a product of those of $\mathcal{M}_1$ and $\mathcal{M}_2$.
- The atomic proposition of $\mathcal{M}$ is a union of of those of $\mathcal{M}_1$ and $\mathcal{M}_2$.

The following theorem shows that L-bisimulation is commutative and associative with respect to approximate synchronization operator.

Theorem

Let $\mathcal{M}_i, i = 1, 2, 3$ be LDLTSs. Given $\beta \in L$. Then

(1) $\sim_{\alpha,\beta} (\mathcal{M}_1 \|_\beta \mathcal{M}_2, \mathcal{M}_2 \|_\beta \mathcal{M}_1) = 1$;

(2) $\sim_L ((\mathcal{M}_1 \|_\beta \mathcal{M}_2) \|_\beta \mathcal{M}_3, \mathcal{M}_1 \|_\beta (\mathcal{M}_2 \|_\beta \mathcal{M}_3)) = 1$;

Logical Characterisation of L-bisimulation

Definition

Given a set of propositions AP and of actions Σ. Let Var be a set of variables. The formulas of the lattice-valued μ-calculus ($L\mu$) are generated as follows:

- if $p \in AP$, then p is a formula.
- A variables x is a formula.
- if $l \in L$, then l is a formula.
- if φ_1 and φ_2 are formulas, then $\neg\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2$ are formulas.
- If $l \in L$, $p \in AP$ and φ is formula, then $p \to l$ and $l \to \varphi$ are formulas.
- If φ is a formula, $d \in \Sigma$, then $\exists \bigcirc_d \varphi$ and $\forall \bigcirc_d \varphi$ are formulas.
- If $x \in Var$ and φ is a formula, then $\mu x.f$ and $\nu x.f$ are formulas, provided that all occurrences of x within φ fall under an even number of negations in φ.

Definition

The truth value of a formula φ of $L\mu$ in a state s of an LDLTS $\mathcal{M}$ and an interpretation ρ, written $[\![\varphi]\!]_\rho^{\mathcal{M}}(s)$, is defined inductively as follows: $\mathcal{L}$ is a strong residuated lattice. The truth value of an $L\mu$ formula φ in an LDLTS $\mathcal{M}$, denoted

$$[\![\varphi]\!]_\rho^{\mathcal{M}} = \bigwedge_{s \in S} (S^0(s) \rightarrow [\![\varphi]\!]_\rho^{\mathcal{M}}(s)).$$

- A variable x is bound in φ if it is in the scope of a quantifier μx or νx; otherwise, it is called free. A formula is closed if all variables are bound.
- If φ is closed, we write $[\![\varphi]\!]^{\mathcal{M}}$ for $[\![\varphi]\!]^{\mathcal{M}}_{\rho}$.
- We also denote by *LHML* the subsets of formulas that do not contain variables *Var*, fixpoints operators, negation operator and implication operators.

The link between L-(bi)simulation and our $L\mu$ semantics for LDLTSs is as follows:

Theorem

Let $\mathcal{M}_i, i = 1, 2$, be two LDLTSs, $\mathcal{R} \in \mathcal{L}(S_1 \times S_2)$ be an L-relation. For all states $s_1 \in S_1$ and $s_2 \in S_2$ and all $L\mu$ closed formulas φ, the following hold.

(1) *If $\mathcal{R}$ is an L-bisimulation relation, then* $\mathcal{R}(s_1, s_2) \leq [\![\varphi]\!]^{\mathcal{M}_2}(s_2) \leftrightarrow [\![\varphi]\!]^{\mathcal{M}_1}(s_1)$.

(2) $\sim_L (\mathcal{M}_1, \mathcal{M}_2) \leq [\![\varphi]\!]^{\mathcal{M}_2} \leftrightarrow [\![\varphi]\!]^{\mathcal{M}_1}$.

For *ops* $\subseteq \{\neg, \mu, \nu\}$, we denote by $L\mu\backslash ops$ the set of formulas that do not employ the operators in *ops*. The following theorem identifies the fragment of the logics that suffices for characterizing the *L*-bisimulation.

Theorem

Let $\mathcal{M}_i$ be two LDLTSs, for every $s_1 \in S_1, s_2 \in S_2$, then There exists a formula $\varphi \in L\mu\backslash\{\neg, \mu, \nu\}$ such that $[\![\varphi]\!]_\rho^{\mathcal{M}_2}(s_2) \leq (s_1 \sim_L s_2)$.

L-bisimulation Quotient Transition Systems

Definition

Let $R \in \mathcal{L}(X \times Y)$. The relation $R^{\exists\exists}, R^{\forall\forall} \in \mathcal{L}(\mathcal{L}(X) \times \mathcal{L}(Y))$ are defined as follows, for every $A \in \mathcal{L}(X), B \in \mathcal{L}(Y)$,

$$R^{\exists\exists}(A, B) = \bigvee_{x \in X} \bigvee_{y \in Y} (A(x) \otimes B(y) \otimes R(x, y)),$$

$$R^{\forall\forall}(A, B) = \bigwedge_{x \in X} (A(x) \to \bigwedge_{y \in Y} (B(y) \otimes R(x, y)))$$

For notation ease, we write $R_a(s, t)$ for $R(s, a, t)$.

Definition

For a given LDLTS $\mathcal{M}$ and its L-bisimulation $\sim_L$. The L-bisimulation quotient transition system $\mathcal{M}/\sim_L$ is defined by

$$\mathcal{M}/\sim_L = (L, S/\sim_L, S^0/\sim_L, \Sigma, AP, R_u, R_o, \mathcal{V}_u, \mathcal{V}_o, \theta),$$

where

- $S/\sim_L = \{[s] \mid s \in S\}$ with $[s](t) = (s \sim_L t)$
- for every $[s] \in S/\sim_L$, $S^0/\sim_L([s]) = S^0(s)$.
- the transition relations R_u and R_o are defined as: for every $[s],[t] \in S/\sim_L$,

$$R_u([s], a, [t]) = R_a^{\exists\exists}([s],[t]),$$
$$R_o([s], a, [t]) = R_a^{\forall\forall}([s],[t]).$$

- The proposition interpretation function $\mathcal{V}_u$ and $\mathcal{V}_o$ is defined as, for every state $[s] \in S/\sim_L$ and atomic proposition $p \in AP$,

$$\mathcal{V}_u([s], p) = \bigvee_{s_1 \in S} ([s](s_1) \otimes \mathcal{V}(s_1, p)),$$
$$\mathcal{V}_o([s], p) = \bigwedge_{s_1 \in S} ([s](s_1) \to \mathcal{V}(s_1, p)).$$

In the sequel, $\mathcal{M}/\sim_L$ is referred to the L-bisimulation quotient of $\mathcal{M}$. For the sake of simplicity, we write $\mathcal{M}/\sim_L$ (resp. $S/\sim_L$) instead of $\mathcal{M}/\sim$ (resp. $S/\sim$).

Theorem

The L-bisimulation quotient transition system defined above is consistent, i.e., for every $[s], [t] \in S/\sim, p \in AP, a \in \Sigma$, *the following assertion holds.*

$$R_o([s], a, [t]) \leq R(s, a, t) \leq R_u([s], a, [t])$$
$$\mathcal{V}_o([s], p) \leq \mathcal{V}(s, p) \leq \mathcal{V}_u([s], p)$$

Definition

Given an L-bisimulation quotient transition $\mathcal{M}/\sim$.The over-approximate semantics and under-approximate semantics of *LHML* are given by the function $[\![\cdot]\!]_e^{\mathcal{M}/\sim}([s])(e = u, o)$, which for each formula φ of *LHML*, a model $\mathcal{M}/\sim$, a state $[s]$ in $\mathcal{M}/\sim$, returns the value of φ at the state $[s]$ of the model $\mathcal{M}/\sim$, defined as follows:

$$
\begin{aligned}
[\![l]\!]_e^{\mathcal{M}/\sim}([s]) &= l \\
[\![p]\!]_e^{\mathcal{M}/\sim}([s]) &= \begin{cases} \mathcal{V}_u([s], p) & \text{if } e = u \\ \mathcal{V}_o([s], p) & \text{if } e = o \end{cases} \\
[\![\varphi_1 \vee \varphi_2]\!]_e^{\mathcal{M}/\sim}([s]) &= \begin{cases} [\![\varphi_1]\!]_u^{\mathcal{M}/\sim}([s]) \vee [\![\varphi_2]\!]_u^{\mathcal{M}/\sim}([s]) & \text{if } e = u \\ [\![\varphi_1]\!]_o^{\mathcal{M}/\sim}([s]) \vee [\![\varphi_2]\!]_o^{\mathcal{M}/\sim}([s]) & \text{if } e = o \end{cases} \\
[\![\varphi_1 \wedge \varphi_2]\!]_e^{\mathcal{M}/\sim}([s]) &= \begin{cases} [\![\varphi_1]\!]_u^{\mathcal{M}/\sim}([s]) \wedge [\![\varphi_2]\!]_u^{\mathcal{M}/\sim}([s]) & \text{if } e = u \\ [\![\varphi_1]\!]_o^{\mathcal{M}/\sim}([s]) \wedge [\![\varphi_2]\!]_o^{\mathcal{M}/\sim}([s]) & \text{if } e = o \end{cases}
\end{aligned}
$$

$$
[\![\exists \bigcirc_d \varphi]\!]_e^{\mathcal{M}/\sim}([s]) = \begin{cases} \bigvee_{[s'] \in S/\sim} \bigvee_{a \in \Sigma} (R_u([s], a, [s']) \otimes \\ \theta(a, d) \otimes [\![\varphi]\!]_u^{\mathcal{M}/\sim}([s'])) & \text{if } e = u \\ \bigvee_{[s'] \in S/\sim} \bigvee_{a \in \Sigma} (R_o([s], a, [s']) \otimes \\ \theta(a, d) \otimes [\![\varphi]\!]_o^{\mathcal{M}/\sim}([s'])) & \text{if } e = o \end{cases}
$$

$$
[\![\forall \bigcirc_d \varphi]\!]_e^{\mathcal{M}/\sim}([s]) = \begin{cases} \bigwedge_{[s'] \in S/\sim} \bigwedge_{a \in \Sigma} (R_o([s], a, [s']) \otimes \\ \theta(a, d) \rightarrow [\![\varphi]\!]_u^{\mathcal{M}/\sim}([s'])) & \text{if } e = u \\ \bigwedge_{[s'] \in S/\sim} \bigwedge_{a \in \Sigma} (R_u([s], a, [s']) \otimes \\ \theta(a, d) \rightarrow [\![\varphi]\!]_o^{\mathcal{M}/\sim}([s'])) & \text{if } e = o \end{cases}
$$

Theorem

Given an LDLTS $\mathcal{M}$ and its L-bisimulation quotient transition system $\mathcal{M}/\sim$, for any formula φ in LHML,

$$[\![\varphi]\!]_o^{\mathcal{M}/\sim}([s]_\sim) \leq [\![\varphi]\!]^{\mathcal{M}}(s) \leq [\![\varphi]\!]_u^{\mathcal{M}/\sim}([s]_\sim).$$

Theorem

Assume $\mathcal{L} \in \mathcal{SRL}$. *Let* φ *be a formula of LHML,* $[s]/\sim$ *a state of* $\mathcal{M}/\sim$. *Given* $[\![\varphi]\!]_o^{\mathcal{M}/\sim}([s]_\sim), [\![\varphi]\!]_u^{\mathcal{M}/\sim}([s]_\sim)$, *the for any* $t \in S$, *we have*

$$[\![\varphi]\!]_o^{\mathcal{M}/\sim}([s]_\sim) \otimes (s \sim_L t) \leq [\![\varphi]\!]^{\mathcal{M}}(t) \leq (s \sim_L t) \rightarrow [\![\varphi]\!]_u^{\mathcal{M}/\sim}([s]_\sim).$$

Our Contribution

- A general Model–Lattice-valued Double Transition Systems(Residuated lattice)
- A general approximate equivalence–L-bisimulation(L-equivalence relation)
- A useful lift–L-bisimulation quotient transition systems

Ongoing and Future Consideration

- Extend trace equivalence to lattice-valued setting, give its logical analysis, analysis its robust properties of lattice-valued trace equivalence. (Finished)
- Translation of many kind of lattice-valued transition systems (LKS,LTS,LDLTS): preservation of Lattice-bisimulation and Lattice-valued Trace equivalence, lattice-valued temporal logic. (Partly Finished)
- Generalize LDLTSs to interval-valued residuated lattice setting, obtain more general model
- To model check based on LDLTSs

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Thank you! & Questions?